3 Discrete time models II

3.1 The multiperiod binary model

Our single step binary model is, of course, inadequate as a model of the evolution of an asset price. One way to think of it is as a model of how prices evolve over a single 'tick' of a clock. As the next level of generality, we extend this model to include several ticks of the clock.

We again think of our market as consisting of just two securities: a bond (representing riskless borrowing) and a stock S. Unlimited amounts of either can be bought and sold without transaction costs, and so on. At every tick of a clock, we are allowed to readjust our portfolio.

As in the single period binary model, we shall suppose that at each tick of the clock, the stock moves from its current value to one of two possible values (depending on its current value). There are 2^i possible states of the stock price after i ticks of the clock, and we think of them as being arranged in a tree as in Figure 4.

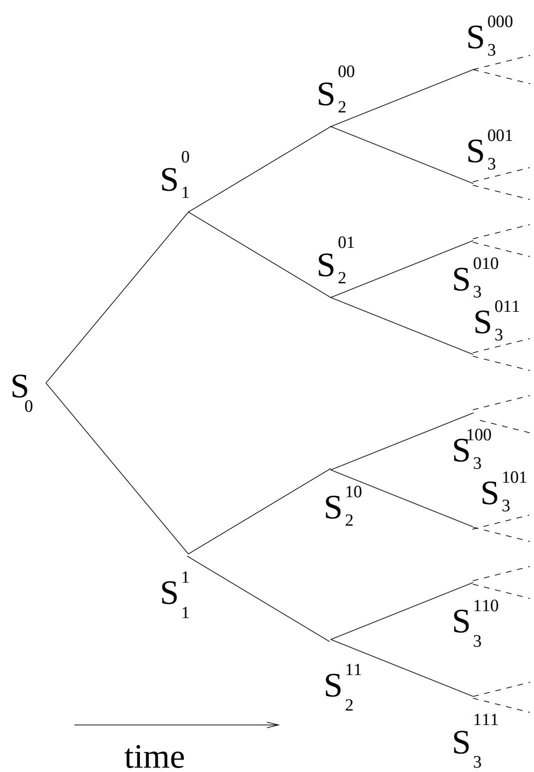

Figure 4.

The bond moves as before, so over the ith tick its value is scaled up by $\exp(r_i)$. We assume that all r_i are known.

Suppose again that I am pricing a European option with strike price K at the maturity time, T. We assume now that $T = n$. The payoff of the option is then $(S_n - K)_+$ at time n.

The idea is to use backward induction on the tree as follows. If we knew the price, S_{n-1}, of the stock after $(n-1)$ ticks, then our previous analysis would tell us the value, C_{n-1}, of the claim at time $(n-1)$. Namely $C_{n-1} = \psi_0^{(n)}\mathbb{E}_{n-1}[C_n]$ where the expectation is with respect to a probability measure for which $S_{n-1} = \psi_0^{(n)}\mathbb{E}_{n-1}[S_n]$ and $\psi_0^{(n)} = e^{-r_n}$. So for each state of the market at time $(n-1)$, I know that I need a portfolio worth C_{n-1} if I am to meet the claim against me at time n. I can now think of C_{n-1} as a claim at time $(n-1)$. In the same way then, if I know S_{n-2}, in order to meet the claim against me at time $(n-1)$, I need to hold a portfolio worth $\psi_0^{(n-1)}\mathbb{E}_{n-2}[C_{n-1}]$, where the expectation is with respect to a measure such that $S_{n-2} = \psi_0^{(n-1)}\mathbb{E}_{n-2}[S_{n-1}]$, and this in turn guarantees that I can exactly meet the claim against me at time n. Continuing in this way, we can successively calculate the cost of a portfolio that, after appropriate readjustment at each tick of the clock, but without any extra input of wealth and without paying dividends, will allow us to exactly meet the claim against us at time n.

The strategy that readjusts the portfolio in this way is said to be a *self-financing* strategy.

The probability measures used at each stage in the above prescribe exactly one probability for each branch in our tree of stock prices. For each vertex of the tree there is a unique path from the vertex through the tree that the stock price could have followed to reach that vertex, and we specify a probability measure on paths by declaring that the probability of such a path is the product of the probabilities on the branches that comprise it.

Let us assume, for simplicity, that the rate of interest is everywhere zero. There is no loss of generality in this – it is equivalent to replacing S_i by the *discounted security price*

$$\tilde{S}_i = \prod_{j=1}^{i} \psi_0^{(j)} S_i = e^{-\sum_1^i r_j} S_i.$$

Our risk neutral probabilities then have the property that $\mathbb{E}[S_k | S_{k-1}] = S_{k-1}$ for each $k = 1, 2, \ldots, n$. In fact much more is true. To illustrate, consider the two-step model in Figure 5.

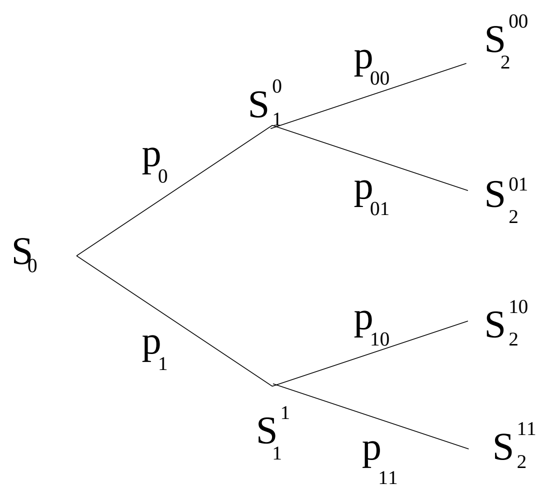

Figure 5.

Using the notation in Figure 5, we have,

$$\begin{aligned}
\mathbb{E}[S_2] &= p_0 p_{00} S_2^{00} + p_0 P_{01} S_2^{01} + p_1 p_{10} S_2^{10} + p_1 p_{11} S_2^{11} \\
&= p_0 \left(p_{00} S_2^{00} + p_{01} S_2^{01} \right) + p_1 \left(p_{10} S_2^{10} + p_{11} S_2^{11} \right) \\
&= p_0 \mathbb{E}\left[S_2 | S_1 = S_1^0 \right] + p_1 \mathbb{E}\left[S_2 | S_1 = S_1^1 \right] \\
&= \mathbb{E}\left[\mathbb{E}\left[S_2 | S_1 \right] \right] \\
&= \mathbb{E}[S_1] = S_0.
\end{aligned}$$

More generally, if $j > i$,

$$\mathbb{E}[S_j | S_i] = S_i.$$

The same argument shows that $C_{n-i} = \mathbb{E}[C_n | S_{n-i}]$, where the expectation is with respect to the probabilities on paths defined above.

The sequence of (discounted) prices, $\{S_i\}_{i=0}^n$, is a *stochastic process* and the probability measure that we have defined has the property that $\mathbb{E}[S_j | S_i] = S_i$. Moreover, in this model, the stock price 'has no memory' so that the movement of the stock over the next tick of the clock is not influenced by the way in which the stock reached its current value and so for $j > i$,

$$\mathbb{E}[S_j | S_0, S_1, \ldots S_i] = S_i.$$

Definition 3.1 *A sequence of random variables (or stochastic process)* $X_0, X_1, \ldots, X_n$ *with* $\mathbb{E}[|X_r|] < \infty$ *for each* r *is a* martingale *if*

$$\mathbb{E}[X_j | X_0, X_1, \ldots X_{j-1}] = X_{j-1} \qquad r = 1, 2, \ldots, n. \tag{3}$$

The idea comes from gambling where if X_r denotes the capital of the gambler at time r then the game is 'fair' only if (3) holds.

Warning

The notion of martingale is really that of a $\mathbb{P}$-martingale. It does not make sense to talk about martingales without specifying a probability measure.

The *'information'* $X_0, X_1, \ldots, X_{r-1}$ is often written $\mathcal{F}_{r-1}$. In a continuous setting we will be a little more careful about the definition, but the idea is the same, $\mathcal{F}_{r-1}$ is the set of events that are 'decidable' by observing the process X_i up to time $r - 1$.

We now recast the results of \$2 in this language.

Suppose that the possible values that the stocks $S_1, \ldots, S_N$ can take on at times $1, 2, 3, \ldots$ are known. We denote by Ω the set of all possible 'paths' that the stock price vector can follow in $\mathbb{R}_+^N$. (We are moving away from the binary model now, but the same argument that allows us to pass from the single period binary model to the multi-period binary model allows us to move from the single period model of Theorems 2.5–2.7 to this multiperiod setting.)

Theorem 2.5 tells us that the absence of arbitrage is equivalent to the existence of a probability measure, $\mathbb{Q}$, on Ω that assigns strictly positive mass to every $\omega \in \Omega$ and such that

$$S_{r-1} = \psi_0^{(r)} \mathbb{E}_{\mathbb{Q}}[S_r | S_{r-1}],$$

where S_i is the vector of stock prices at time r.

If, as above, we consider the *discounted* stock prices, then

$$\mathbb{E}_{\mathbb{Q}}[\tilde{S}_r|\tilde{S}_1, \dots, \tilde{S}_{r-1}] = \tilde{S}_{r-1}.$$

In other words, the discounted stock price vector is a $\mathbb{Q}$-*martingale.*

Definition 3.2 *Two probability measures* $\mathbb{P}$ *and* $\mathbb{Q}$ *on a space* Ω *are said to be* equivalent *if for events* $A \subseteq \Omega$

$$\mathbb{Q}(A) = 0 \qquad \textit{if and only if} \qquad \mathbb{P}(A) = 0.$$

Suppose then that we have a market model in which the stock price vector can follow one of a finite number of paths Ω through $\mathbb{R}^n_+$. We may even have our own belief as to how the price will evolve, encoded in a probability measure, $\mathbb{P}$, on Ω.

Theorems 2.5 and 2.7 combine to say

Theorem 3.3 *There is no arbitrage if and only if there is an equivalent martingale measure* $\mathbb{Q}$. *That is, there is a measure,* $\mathbb{Q}$, *equivalent to* $\mathbb{P}$, *such that the discounted price process is a* $\mathbb{Q}$-*martingale.*

In that case, the market price of an attainable claim C *(to be delivered at time* n*) at time zero is unique and is given by*

$$\mathbb{E}_{\mathbb{Q}}[\psi_0 C],$$

where $\psi_o = \prod_1^n \psi_0^{(i)}$ *is the discount factor over* n *periods.*

This fundamental theorem will have essentially the same statement in the continous setting.

3.2 The Cox Ross Rubinstein model

Definition 3.4 *The* Cox Ross Rubinstein (CRR) model *is the name given to the special case of the multiperiod binary model in which in each time interval the stock price moves from its current value,* S, *to one of* Su, Sd, *where* u *and* d *are fixed constants with* $d < e^{r\Delta T} < u$.

Notice that in this case the tree *recombines.* This is often refered to as the binomial model. At time k, there are k possible values that the price can take and

$$\mathbb{P}[S_k = S_0 u^j d^{k-j}] = \begin{pmatrix} k \\ j \end{pmatrix} p^j (1-p)^{k-j}.$$

The recombining of the tree makes this highly numerically efficient.

The values of u and d must be calibrated with the market. The usual assumption is that $u = 1/d$ and p can then be determined by risk neutrality as

$$p = \frac{e^{r\Delta t} - d}{u - d}.$$

Finally, u is fitted using the *variance* of the stock price.