Typical distances in ultrasmall random networks

(1) Ultrasmall networks: a reminder

(2) Typical distances in configuration networks

(3) Typical distances in preferential attachment networks

(4) A model-free approach to lower bounds

Ultrasmall networks: a reminder

Given $\mathcal{G}_N$ we let $d(\cdot\,,\,\cdot)$ be the graph distance of two vertices, i.e. the length of the shortest path between them. Picking two vertices V, W independently, uniformly from the giant component, we say the network is ultrasmall if

$$\lim_{N\to\infty} \frac{d(V,W)}{\log\log N} = c > 0 \quad \text{in probability.}$$

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Note: We don't expect to gain significant insight into the case $\tau \leq 2$ as our models are undefined or degenerate in this case.

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Note: We expect results about the diameter of the giant component to depend considerably on the model details, and thefore to be of less interest.

Typical distances in configuration networks

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Networks with fixed degree sequence use a sequence $D_1, D_2, \ldots$ of iid random variables with

$$\mathbb{P}\{D_1 > x\} = x^{1-\tau}(c + o(1)) \quad \text{as } x \uparrow \infty.$$

Given $D_1, \ldots, D_N$ we construct the network $\mathcal{G}_N$ by attaching D_i half-edges to the vertex labelled i, and matching them at random.

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Theorem (van der Hofstad and Hooghiemstra 2008)

The networks with given fixed degree sequence are ultrasmall if and only if

$$2 < \tau < 3.$$

Moreover, for independent, uniformly chosen vertices V and W in the giant component of $\mathcal{G}_N$, we have

$$\lim_{N\to\infty} \frac{d(V,W)}{\log\log N} = \frac{2}{-\log(\tau-2)} \qquad \text{in probability.}$$

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Remark: The limit depends only on τ.

Typical distances in configuration networks

Conditionally Poissonian random graphs are based on drawing an iid sequence $\Lambda_1, \Lambda_2, \ldots$ of positive fitness values with

$$\mathbb{P}\{\Lambda_1 > x\} = x^{1-\tau}(c + o(1)) \quad \text{as } x \uparrow \infty$$

Conditional on this sequence, we independently connect vertices n, m in $\mathcal{G}_N$ by a Poissonian number of vertices with mean

$$\frac{\Lambda_n \Lambda_m}{\sum_{k=1}^N \Lambda_k}.$$

The conditionally Poissonian random graph is scale-free with power-law exponent τ.

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The networks with heavy-tailed fitness distribution are ultrasmall if and only if

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Moreover, for independent, uniformly chosen vertices V and W in the giant component of $\mathcal{G}_N$, we have

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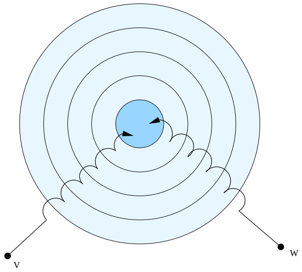
Remark: The limit is the same as in the first example.

Typical distances in configuration networks

At least heuristically we have some structural insight into typical shortest paths in ultrasmall configuration networks:

- typical vertices in the giant component can be connected with a few steps to a core of the network;
- within this core there is a hierarchy of layers of nodes with increasing connectivity and at the top a small inner core of highly connected nodes with very small diameter;
- a shortest path inside the core moves from one layer to the next until the inner core is reached, and then climbing down again until a vertex in the lowest layer of the core is again connected to a typical vertex.



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The jth layer consists of vertices with degree k_j where

$$\log k_j \approx (\tau - 2)^{-j}$$

and there are about

$$\frac{\log \log N}{-\log(\tau - 2)}$$

layers. The graph distance of two randomly chosen vertices in the giant component is therefore

$$\big(2 + o(1)\big) \frac{\log \log N}{-\log(\tau - 2)}.$$

Typical distances in preferential attachment networks

We first look at preferential attachment networks with fixed outdegree, given by parameters $\delta > -m$ where $m \geq 2$ is an integer.

- $\mathcal{G}_1$ consists of a single vertex with m self loops.
- Given $\mathcal{G}_N$, we insert one new vertex and then successively insert m edges connecting the new vertex to vertex $n \leq N$ with probability

$$\sim \text{ (degree of vertex } n) + \delta$$

or to itself with probability

$$\sim \text{ (current degree)} + \frac{\delta}{m}.$$

This network is scale-free with power-law exponent

$$\tau = 3 + \frac{\delta}{m},$$

and we expect it to be ultrasmall iff $\delta < 0$.

Typical distances in preferential attachment networks

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Theorem

For the preferential attachment model with $\delta = 0$ and independent, uniformly chosen vertices V and W in the giant component of $\mathcal{G}_N$,

$$\lim_{N\to\infty} d(V,W)\,\frac{\log\log N}{\log N} = 1 \qquad \text{in probability.}$$

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For the preferential attachment model with $\delta < 0$ and independent, uniformly chosen vertices V and W in the giant component of $\mathcal{G}_N$,

$$\limsup_{N\to\infty} \frac{d(V,W)}{\log\log N} \leq \frac{4}{-\log(\tau-2)} \qquad \text{in probability.}$$

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Problems:

- Find a lower bound and thereby verify ultrasmallness if $2 < \tau < 3$.
- Identify the correct limit. Is this limit universal?
- Find a similar result for preferential attachment networks with variable outdegree.

A model-free approach to lower bounds

Our first result is based on the following assumption.

Assumption $\mathrm{PA}(\gamma)$

There exists κ such that, for all pairwise distinct vertices $v_0, \ldots, v_\ell$ in $\mathcal{G}_N$,

$$\mathbb{P}\Big\{ v_0 \leftrightarrow v_1 \leftrightarrow v_2 \leftrightarrow \cdots \leftrightarrow v_\ell \Big\} \leq \prod_{k=1}^{\ell} \kappa \left(v_{k-1} \wedge v_k\right)^{-\gamma} \left(v_{k-1} \vee v_k\right)^{\gamma-1}.$$

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For preferential attachment models with fixed or variable outdegree and power law exponent τ, we can easily verify that

$$\gamma > (\tau - 1)^{-1} \quad \Longrightarrow \quad \text{Assumption } \mathrm{PA}(\gamma).$$

Hence we expect networks to be ultrasmall if $\mathrm{PA}(\gamma)$ holds for $\frac{1}{2} < \gamma < 1$.

A model-free approach to lower bounds

Theorem 9

Suppose $\mathcal{G}_N$ satisfies Assumption $\mathrm{PA}(\gamma)$ for some $\frac{1}{2} < \gamma < 1$. For random vertices V and W chosen independently and uniformly from $\mathcal{G}_N$, we have

$$d(V, W) \geq 4\,\frac{\log\log N}{\log(\frac{\gamma}{1-\gamma})} + \mathcal{O}(1) \qquad \text{with high probability.}$$

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Corollary 1

The preferential attachment model with fixed outdegree and parameters $\delta > -m$ is ultrasmall if and only if $\delta < 0$ or, equivalently $2 < \tau < 3$. Moreover, for independent, uniformly chosen vertices V and W in the giant component of $\mathcal{G}_N$, we have

$$\lim_{N\to\infty} \frac{d(V,W)}{\log\log N} = \frac{4}{-\log(\tau-2)} \qquad \text{in probability.}$$

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Corollary 2

The preferential attachment model with variable outdegree and attachment rule f is ultrasmall if and only if

$$\gamma := \lim_{n\to\infty} \frac{f(n)}{n} > \frac{1}{2}$$

or, equivalently $2 < \tau < 3$. Moreover, for independent, uniformly chosen vertices V and W in the giant component of $\mathcal{G}_N$, we have

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We now briefly sketch how assumption $\mathrm{PA}(\gamma+\epsilon)$ can be verified for the preferential attachment model with variable outdegree.

A model-free approach to lower bounds

For $v < w$, all events $\{v \leftrightarrow w\}$ with different values of v are independent. Hence $\mathbb{P}\{v_0 \leftrightarrow \cdots \leftrightarrow v_n\}$ can be decomposed into factors of the form $\mathbb{P}\{v \leftrightarrow w\}$ and factors of the form $\mathbb{P}\{u \leftrightarrow v \leftrightarrow w\}$ for $v < u, w$.

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Denoting by $Z[n, N]$ the indegree of vertex n in $\mathcal{G}_N$ we get, for $v < w$,

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For sufficiently large v the increments of f are bounded by $\gamma + \epsilon$, and hence

$$Y_n = f(Z[v, n]) \prod_{i=v}^{n-1} \left(1 + \frac{\gamma + \epsilon}{i}\right)^{-1}$$

defines a supermartingale. This implies that

$$\mathbb{E}f(Z[v, w-1]) \leq \kappa w^{\gamma+\epsilon} v^{-\gamma-\epsilon}$$

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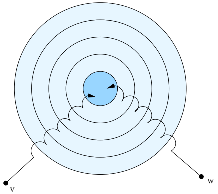
A similar argument can be used to estimate factors of the second form.

A model-free approach to lower bounds

Recall the structural insight into typical shortest paths in ultrasmall configuration networks.



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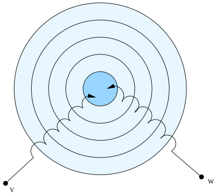

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Other than in models of configuration type, in models of preferential attachment type a high degree of a vertex does not increase its connectivity to all other vertices but only to those introduced late into the system, which are typically outside the core.

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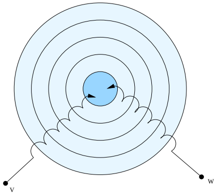

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Other than in models of configuration type, in models of preferential attachment type a high degree of a vertex does not increase its connectivity to all other vertices but only to those introduced late into the system, which are typically outside the core.

Therefore a path cannot move directly from one layer to another in one step, but it requires two steps: The paths move from one layer to a young vertex and from there back into the next higher layer. The distance of two typical vertices is therefore increased by a factor of two.

A model-free approach to lower bounds

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Both models of configuration type we considered (and more) satisfy $\mathrm{CM}(\gamma)$ for all $\gamma > (\tau - 1)^{-1}$ and we obtain a lower bound from the following theorem.

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Theorem 10

Suppose $\mathcal{G}_N$ satisfies Assumption $\mathrm{CM}(\gamma)$ for some $\frac{1}{2} < \gamma < 1$. For random vertices V and W chosen independently and uniformly from $\mathcal{G}_N$, we have

$$d(V,W) \geq 2\,\frac{\log\log N}{\log(\frac{\gamma}{1-\gamma})} + \mathcal{O}(1) \qquad \text{with high probability.}$$

A model-free approach to lower bounds

Summary

The results suggest that the ultrasmall networks can be divided into two universality classes. In networks of preferential attachment type typical vertices have twice the distance compared to networks of configuration type. There is also a different structure to shortest paths in the network, with paths alternating between young and old vertices in the case of preferential attachment networks.

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In real networks, however, this effect is hard to establish, not least because of the slow growth of $\log\log N$. It also seems that it is often overruled by effects not represented in our simple models. For example, in the mathematicians collaboration graph by the effect that the number of authors per paper has increased significantly over the past 50 years, and that mathematicians have a limited period of activity.

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I would however still uphold the claim that this is a good example that rigorous mathematical analysis has identified an interesting effect about networks, that can not be identified by simulation or other nonrigorous methods.

A model-free approach to lower bounds

A small selection of references:

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