Scaling limits of anisotropic random growth models

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Overview

1. Generalised HL(0) clusters
2. Loewner chains driven by measures
3. A shape theorem for anisotropic HL(0) clusters
4. The evolution of harmonic measure on the cluster boundary

Conformal mapping representation of a cluster

Let D_0 denote the exterior unit disk in the complex plane $\mathbb{C}$. Let $K_0 = \mathbb{C} \setminus D_0$ be the closed unit disk. Consider a simply connected set $D_1 \subset D_0$, such that $P = D_1^c \setminus K_0$ has diameter $d \in (0, 1]$ and $1 \in \overline{P}$. The set P models an incoming particle, which is attached to the unit disk at 1. We use the unique conformal mapping $f_P : D_0 \to D_1$ as a mathematical description of the particle.

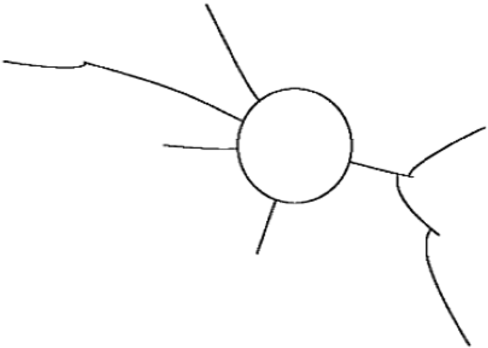
Conformal mapping representation of single particle

Let $P_1, P_2, \ldots$ be a sequence of particles with $\mathrm{diam}(P_j) = d_j$. Let $\theta_1, \theta_2, \ldots$ be a sequence of angles. Define rotated copies $f_{P_j}^{\theta_j}(z)$ of the maps $\{f_{P_j}\}$ so that $f_{P_j}^{\theta_j}(D_0) = e^{i\theta_j} f_{P_j}(D_0)$. Take $\Phi_0(z) = z$, and recursively define

$$\Phi_n(z) = \Phi_{n-1} \circ f_{P_n}^{\theta_n}(z), \quad n = 1, 2, \ldots.$$

This generates a sequence of conformal maps $\Phi_n : D_0 \to D_n = \mathbb{C} \setminus K_n$, where $K_{n-1} \subset K_n$ are growing compact sets, or clusters.

The slit model after a few arrivals with $d = 1$

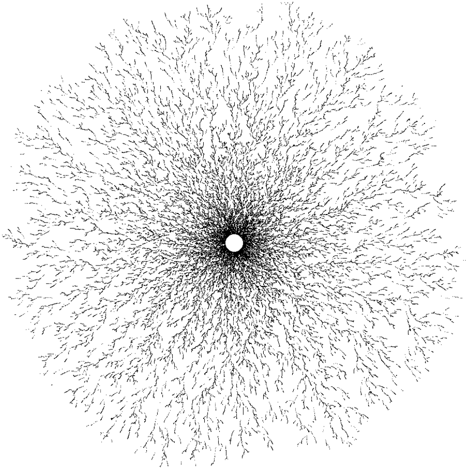


Generalised Hastings-Levitov clusters

By choosing the sequences $\{\theta_j\}$ and $\{d_j\}$ in different ways, it is possible to describe a wide class of growth models.

In the Hastings-Levitov family of models $\mathrm{HL}(\alpha)$, $\alpha \in [0, 2]$, the θ_j are chosen to be independent uniform random variables on the unit circle which corresponds to the attachment point at the nth step being distributed according to harmonic measure at infinity for K_{n-1}. The particles are usually taken to be "slits" with diameters taken as $d_j = d/|\Phi'_{j-1}(e^{i\theta_j})|^{\alpha/2}$. Heuristically, the case $\alpha = 1$ corresponds to the Eden model (biological cell growth) and the case $\alpha = 2$ is a candidate for off-lattice DLA.

HL(0) cluster with 25000 particles for $d = 0.02$



Anisotropic Hastings-Levitov model

Anisotropic Hastings-Levitov, $\mathrm{AHL}(\nu)$, is a variant of the $\mathrm{HL}(0)$ model in which $\theta_1, \theta_2, \ldots$ are i.i.d. random variables on the unit circle with common law ν and $d_j = d$.

Models can be further generalised by allowing $P_1, P_2, \ldots$ to be chosen randomly from a class of suitable shapes, even with $d_1, d_2, \ldots$ i.i.d. random variables (independent of $\{\theta_j\}$) satisfying certain conditions, however our results are not sensitive to these changes.

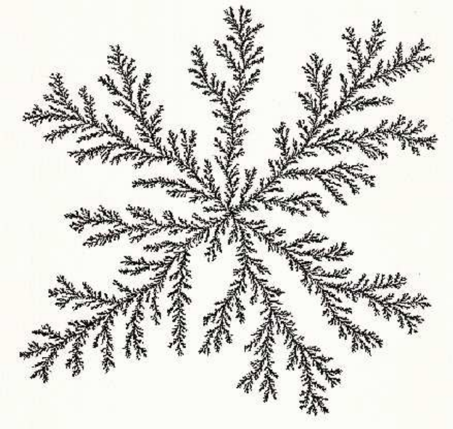
Motivation for anisotropic models

The use of more general distributions for the angles is a way of introducing localization in the growth, such as can be observed in actual DLA.

Anisotropic versions of DLA can be used to model natural processes such as the formation of hoar frost and it is suggested that anisotropic Hastings-Levitov models may provide a description for the growth of bacterial colonies where the concentration of nutrients is directional.

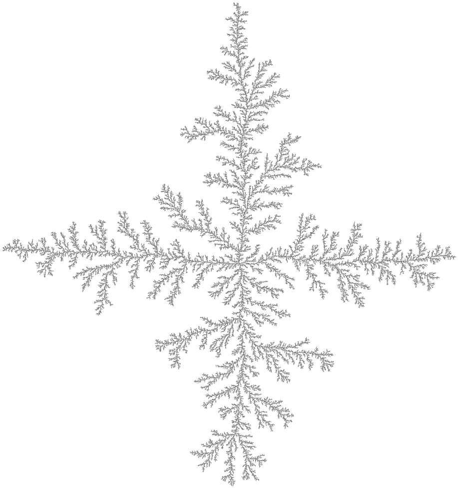
Simulations suggest that anisotropic Hastings-Levitov clusters show less variation as α changes, than isotropic models.

A DLA cluster of size 100 million



Simulation due to Henry Kaufman (Yale)

A DLA cluster on the square lattice of size 4096



Simulation due to Vincent Beffara (ENS Lyon)

Loewner chains

A general way to describe growing (random or deterministic) compact sets is to use Loewner chains.

A decreasing Loewner chain is a family of conformal mappings

$$f_t : D_0 \to \mathbb{C} \setminus K_t, \quad \infty \mapsto \infty, \quad f_t'(\infty) > 0,$$

onto the complements of a growing family of compact sets, called hulls, with

$$K_{t_1} \subset K_{t_2} \quad \text{for} \quad t_1 < t_2.$$

We always take K_0 to be the closed unit disk. The capacity of each K_t is given by

$$\mathrm{cap}(K_t) = \lim_{z \to \infty} \frac{f_t(z)}{z}.$$

Loewner chains driven by probability measures

Let $\mathcal{P} = \mathcal{P}(\mathbb{T})$ denote the class of probability measures on the unit circle $\mathbb{T}$. Under some natural assumptions on the function $t \mapsto \mathrm{cap}(K_t)$, such a chain can be parametrized in terms of families $\{\mu_t\}_{t \geq 0}$, $\mu_t \in \mathcal{P}(\mathbb{T})$.

More precisely, the conformal mappings f_t satisfy the Loewner-Kufarev equation

$$\partial_t f_t(z) = z f_t'(z) \int_{\mathbb{T}} \frac{z+\zeta}{z-\zeta} d\mu_t(\zeta), \tag{1}$$

with initial condition $f_0(z) = z$. Conversely, for any family $\{\mu_t\}_{t \geq 0}$, $\mu_t \in \mathcal{P}(\mathbb{T})$, the solution to (1) exists and is a Loewner chain.

Point masses

In the case of pure point masses

$$\mu_t = \delta_{\xi(t)},$$

where $|\xi(t)| = 1$, the Loewner-Kufarev equation reduces to the equation

$$\partial_t f_t(z) = z f_t'(z) \frac{z + \xi(t)}{z - \xi(t)},$$

which was originally introduced by Loewner in 1923. The function $\xi(t)$ is usually called the driving function.

Slit mappings

The choice $\xi(t) = 1$ produces as solutions the basic slit mappings $f_{d(t)} : D_0 \to D_0 \setminus (1, 1 + d(t)]$, with slit lengths $d(t)$ given by the explicit formula

$$d(t) = 2e^t(1 - \sqrt{1 - e^{-t}}) - 2. \tag{2}$$

We can recover (the slit version of) the $\mathrm{HL}(0)$ mappings Φ_n by driving the Loewner equation with a non-constant point mass at

$$\xi(t) = \exp\left(i \sum_{j=1}^{n} \theta_j \chi_{[T_{j-1}, T_j]}(t) \right),$$

where the times T_j relate to the slit lengths d via the formula (2).

General particle mappings

For $k = 1, \ldots, n$, set

$$T_k = k \log \mathrm{cap}(K_0 \cup P),$$

and let $\xi_k(t)$, $t \in [T_{k-1}, T_k)$, be the (rotated) driving function for the particle P_k. Set

$$\xi^n(t) = \exp\left(i \sum_{k=1}^{n} \chi_{[T_{k-1}, T_k)}(t)\, \xi_k(t) \right).$$

Then $\delta_{\xi^n(t)}$ is the measure that drives the evolution of the AHL clusters. That is, the mapping Φ_n is the solution to the Loewner-Kufarev equation

$$\partial_t f_t(z) = z f_t'(z) \int_{\mathbb{T}} \frac{z+\zeta}{z-\zeta} d\delta_{\xi^n(t)}(\zeta)$$

with $f_0(z) = z$, evaluated at time $t = T_n$.

Absolutely continuous driving measures

Choosing absolutely continuous driving measures

$$d\mu_t = h_t(\zeta)|d\zeta|$$

results in the growth of the clusters no longer being concentrated at a single point. In the simplest case $d\mu_t(\zeta) = |d\zeta|/2\pi$, the Loewner-Kufarev equation reduces to

$$\partial_t f_t(z) = z f_t'(z),$$

and we see that $f_t(z) = e^t z$, so that $K_t = e^t K_0$. Absolutely continuous driving measures arise naturally as limits in connection with the anisotropic $\mathrm{HL}(0)$ clusters.

Continuity properties of the Loewner equation

Our goal is to describe the macroscopic shape of the anisotropic HL clusters in the limit where the particle sizes converge to zero. In order to do this, we need the solutions to the Loewner-Kufarev equation (1) to be "close" at time T if the driving measures are "close" in some suitable sense.

Theorem

Let $0 < T < \infty$. Let $\mu^n = \{\mu^n_t\}_{t \geq 0}$, $n = 1, 2, \dots$, and $\mu = \{\mu_t\}_{t \geq 0}$ be families of measures in $\mathcal{P}$. Let m denote Lebesgue measure on $[0, \infty)$, and suppose that the measures $\mu^n_t \times m$ converge weakly on $S = \mathbb{T} \times [0, T]$ to the measure $\mu_t \times m$ as $n \to \infty$.

Then the solutions $\{f^n_T\}$ to (1) corresponding to the sequence $\{\mu^n\}$ converge to f_T, the solution corresponding to μ, uniformly on

The shape theorem

For fixed $T \in (0, \infty)$, set $T_n = n \log \mathrm{cap}(K_0 \cup P)$. Then by an appropriate version of the strong law of large numbers it can be shown that $\delta_{\xi^n(t)} \times m_{[0,T_n]}$ converges to $\nu \times m_{[0,T]}$ as $d \to 0$ with respect to the weak topology.

Therefore, if

$$\Phi_n = f_{P_1}^{\theta_1} \circ \cdots \circ f_{P_n}^{\theta_n},$$

then Φ_n converges to Φ uniformly on compacts almost surely as $d \to 0$, where Φ denotes the solution to the Loewner-Kufarev equation driven by the measures $\{\nu_t\}_{t \geq 0} = \{\nu\}_{t \geq 0}$ and evaluated at time T.

Angles chosen in an interval

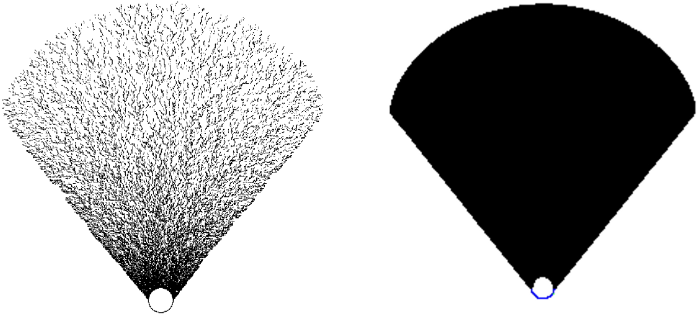
For $\eta \in (0, 1]$, let θ_j be chosen uniformly in $[0, \eta]$. Then

$$d\nu(e^{2\pi i x}) = \frac{\chi_{[0,\eta]}(x)dx}{\eta}.$$

The clusters converge to the hulls of the Loewner chain described by the equation

$$\partial_t f_t(z) = z f_t'(z) \left(1 + \frac{2}{\eta} \arctan\left[\frac{e^{i\pi\eta}\sin(\pi\eta)}{z - e^{i\pi\eta}\cos(\pi\eta)}\right]\right).$$

The slit model on the half circle



Simulation of AHL(ν) and limiting Loewner hull, for $d = 0.02$ after 25000 arrivals, corresponding to $d\nu(e^{2\pi ix}) = 2\chi_{[0,1/2]}(x)dx$.

Angles chosen from a density with *m*-fold symmetry

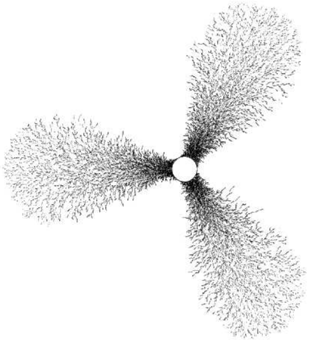
For fixed $m \in \mathbb{N}$, choose θ_j distributed according to the density

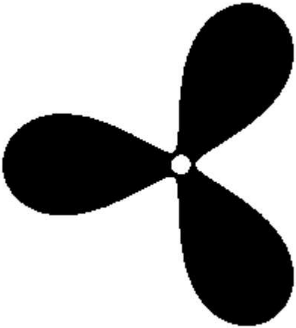
$$d\nu(e^{2\pi i x}) = 2\sin^2(m\pi x)dx.$$

The clusters converge to the hulls of the Loewner chain described by the equation

$$\partial_t f_t(z) = z f_t'(z)\left(1 - \frac{1}{z^m}\right).$$

The slit model for a measure with 3-fold symmetry





Simulation of AHL(ν) and limiting Loewner hull, for $d = 0.02$ after 25000 arrivals, corresponding to $d\nu(e^{2\pi ix}) = 2\sin^2(3\pi x)dx$.

The evolution of harmonic measure on the cluster boundary

For the mapping associated to a particle P, write g_P for the inverse mapping from $D_1 \to D_0$. There exists a unique γ_P that restricts to a continuous map from the interval $(0,1)$ to itself, such that

$$g_P(e^{2\pi i x}) = e^{2\pi i \gamma_P(x)}, \quad x \in (0,1).$$

Set $\Gamma_n = g_{P_n} \circ \cdots \circ g_{P_1}$, where $g_{P_n} = (f_{P_n}^{\theta_n})^{-1}$, so that $\Gamma_n : D_n \to D_0$. The mappings Γ_n induce a flow on the unit circle and this flow describes the evolution of the harmonic measure on the cluster boundary, as particles are added to the cluster.

The fluid limit of the process

Suppose that particles are added at rate $\log \mathrm{cap}(K_0 \cup P)$. Let X be a flow map corresponding to a lifting of Γ onto the real line. Let ϕ be the flow map giving the solution to the deterministic ordinary differential equation

$$\dot{\phi}_{(s,t]}(x) = H[h_\nu](\phi_{(s,t]}(x)), \quad \phi_{(s,s]}(x) = x;$$

where

$$H[h_\nu](\xi) = \text{p.v. } \frac{1}{2\pi} \int_0^1 \cot(\pi(\xi - z)) h_\nu(z) dz$$

is the Hilbert transform of the measure $d\nu = h_\nu dx$.

Then, as $d \to 0$, X converges to ϕ in probability (as flow maps).

Evolution of harmonic measure

Let $x_1, \dots, x_n$ be a positively oriented set of points in $\mathbb{R}/\mathbb{Z}$ and set $x_0 = x_n$. Set $K_t = K_{\lfloor \log \mathrm{cap}(K_0 \cup P)^{-1} t \rfloor}$. For $k = 1, \dots, n$, write ω_t^k for the harmonic measure in K_t of the boundary segment of all fingers in K_t attached between x_{k-1} and x_k. Then, in the limit $d \to 0$, $(\omega_t^1, \dots, \omega_t^n)_{t \geq 0}$ converges weakly in $D([0,\infty), [0,1]^n)$ to $(\phi_{(0,t]}(x_1) - \phi_{(0,t]}(x_0), \dots, \phi_{(0,t]}(x_n) - \phi_{(0,t]}(x_{n-1}))_{t \geq 0}$.

A geometric consequence of this result is that the number of infinite fingers of the cluster converges to the number of stable equilibria of the ordinary differential equation $\dot{x}_t = b(x_t)$, and the positions at which these fingers are rooted to the unit disk converge to the unstable equilibria of the ODE.

Stochastic fluctuations about the limit

For fixed $(s, x) \in [0, \infty) \times \mathbb{R}$, define

$$Z_t^P = (\log \operatorname{cap}(K_0 \cup P)\rho(P))^{1/2}(X_{(s,t]}(x) - \phi_{(s,t]}(x))$$

and let Z_t be the solution to the linear stochastic differential equation

$$dZ_t = \sqrt{h_\nu(\phi_{(s,t]}(x))}dB_t + b'(\phi_{(s,t]}(x))Z_t dt, \quad t \geq s,$$

starting from $Z_s = 0$, where B_t is a standard Brownian motion. Then, as $d \to 0$, the processes $Z_t^P \to Z_t$ in distribution.

Note that if $\phi_{(s,t]}(x)$ stays off the support of h_ν, then $Z_t = 0$ for all $t \geq s$. Also observe that in the case where ν is the uniform measure on the unit circle, $(\log \operatorname{cap}(K_0 \cup P)\rho(P))^{1/2}(X_{(s,t]}(x) - x)_{t \geq s}$ converges to standard Brownian motion, starting from 0 at time s.

Angles chosen in an interval

For

$$d\nu(e^{2\pi ix}) = \frac{\chi_{[0,\eta]}(x)dx}{\eta},$$

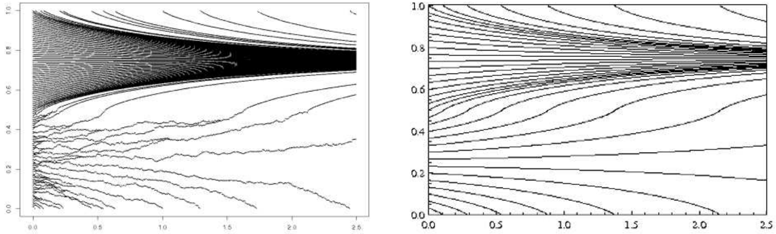
the boundary flow converges to the solution to the ordinary differential equation

$$\dot{\phi}_{(s,t]}(x) = \frac{1}{2\pi^2\eta} \log \left| \frac{\sin(\pi\phi_{(s,t)}(x))}{\sin(\pi(\phi_{(s,t]}(x) - \eta))} \right|$$

with $\phi_{(s,s]}(x) = x$. In the special case $\eta = 1/2$, we obtain the equation

$$\dot{\phi}_{(s,t]} = \frac{1}{\pi^2} \log |\tan(\pi\phi_{(s,t]}(x))|.$$

The slit model on the half circle



Simulation of evolution of harmonic measure on the boundary of AHL(ν) and limiting ODE, for $d = 0.02$ after 25000 arrivals, corresponding to $d\nu(e^{2\pi ix}) = 2\chi_{[0,1/2]}(x)dx$.

Note the absence of random fluctuations in the region $(1/2, 1)$

Angles chosen from a density with *m*-fold symmetry

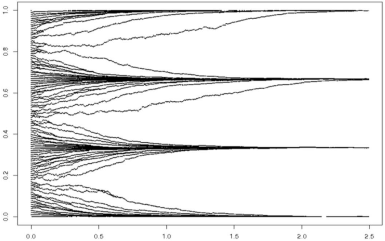
For fixed $m \in \mathbb{N}$, and

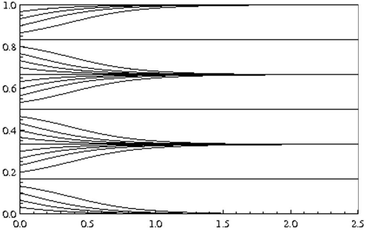
$$d\nu(e^{2\pi ix}) = 2\sin^2(m\pi x)dx$$

the boundary flow converges to the solution to the ordinary differential equation

$$\dot{\phi}_{(s,t]}(x) = -\frac{1}{2\pi}\sin(2\pi m\phi_{(s,t]}(x)), \quad \phi_{(s,s]}(x) = x.$$

The slit model for a measure with 3-fold symmetry





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