IMS/Bernoulli World Congress in Probability and Statistics,
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Scaling limits of critical random trees and graphs

Christina Goldschmidt

Department of Statistics and Lady Margaret Hall

EPSRC
Engineering and Physical Sciences Research Council


Louigi Addario-Berry


Nicolas Broutin

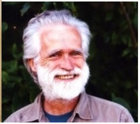

Marie Albenque

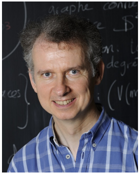

Bénédicte Haas


Grégory Miermont


James Martin


David Aldous


Jim Pitman


Jean-François Le Gall


Jean Bertoin


James Norris

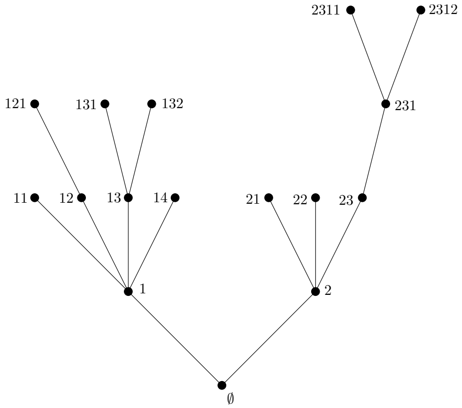

Alison Etheridge

PART I: RANDOM TREES

[Based on work of Aldous, Duquesne, Le Gall, Le Jan, . . .]

Galton–Watson trees

Consider a Galton–Watson branching process with offspring distribution $p = (p_k)_{k \geq 0}$ such that $p_0 + p_1 < 1$. We may associate with it a family tree T.



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Motivation: many natural combinatorial models of random trees may be recovered by taking specific offspring distributions, for example,

- Poisson(1): uniform labelled trees
- Geometric(1/2): uniform plane/ordered trees
- $p_0 = p_2 = 1/2$: uniform (complete) binary trees (n odd).

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Moreover, such trees turn up as component parts of other more complicated graph structures of interest e.g. random planar maps.

Functional encoding

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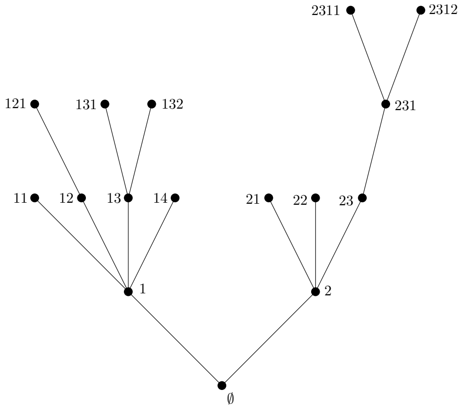
Fix a tree T with $|T| = n$. Let $v(i), 0 \leq i \leq n-1$ be the vertex-labels in lexicographic order and write $d(u, v)$ for the graph distance between two vertices in the tree.

Height process

Let $G(k) = d(v(0), v(k))$ for $0 \leq k \leq n-1$, the generation of vertex $v(k)$.



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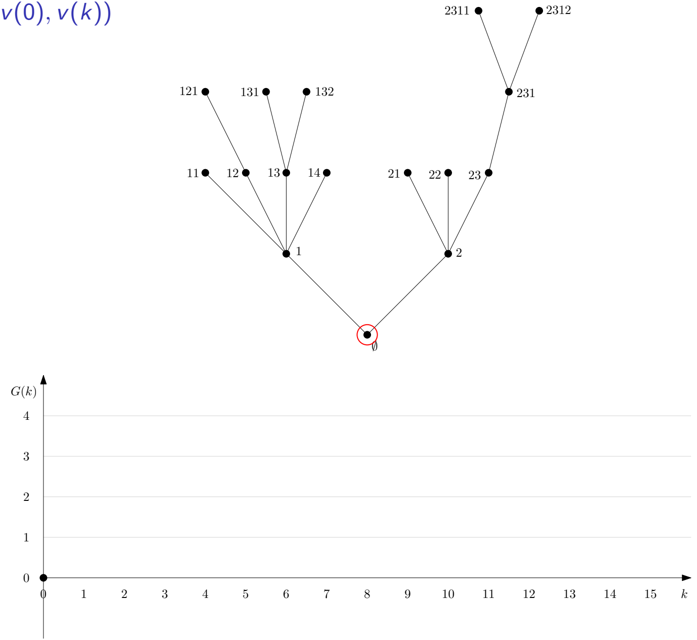

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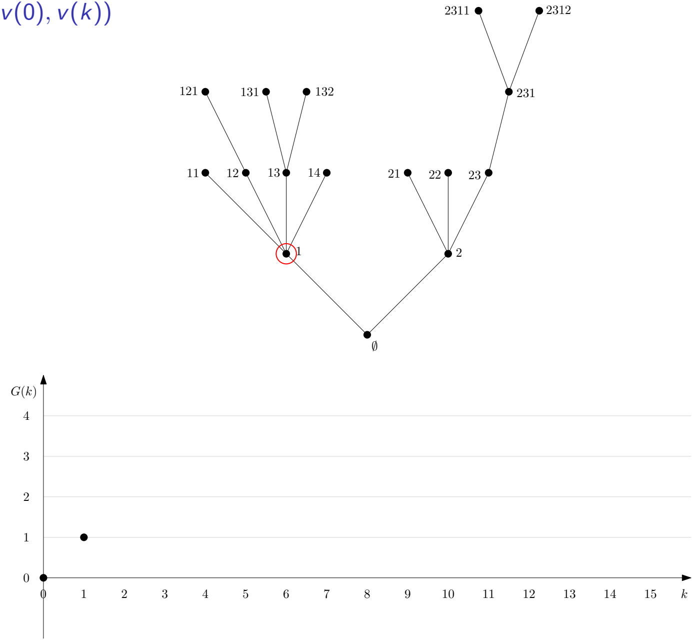

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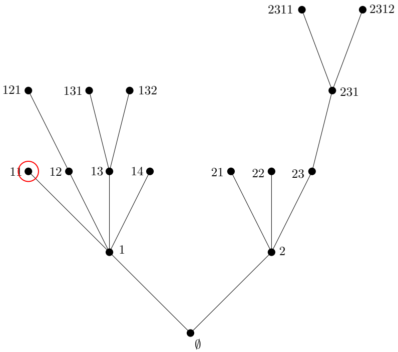

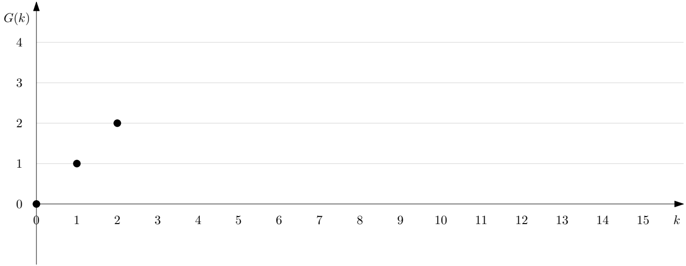

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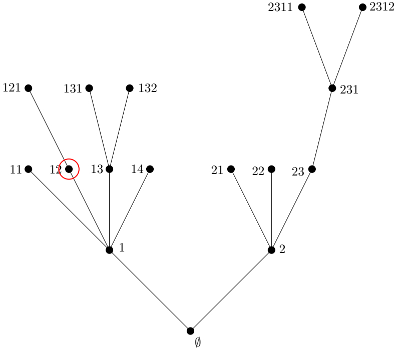

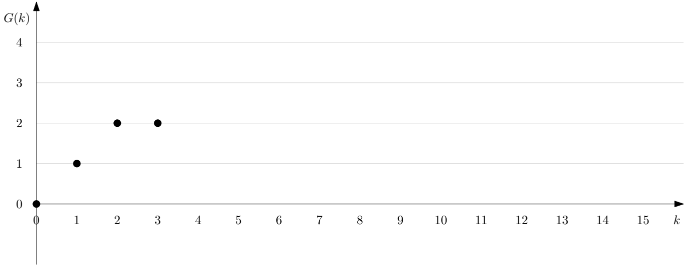

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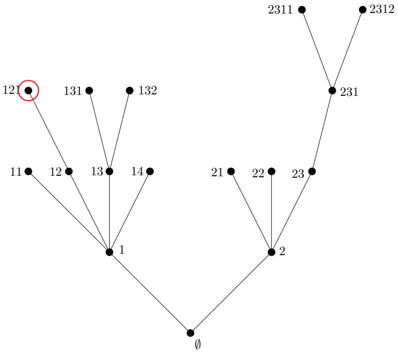

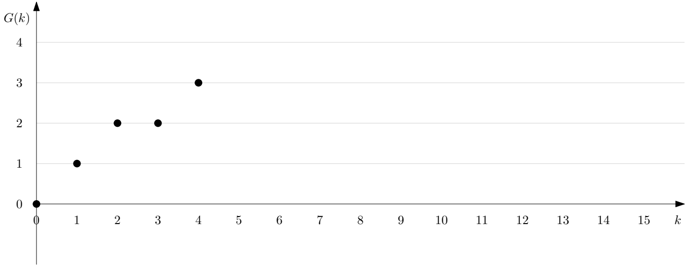

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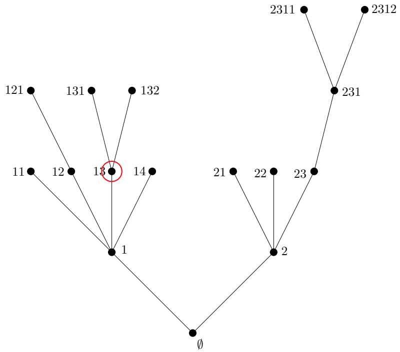

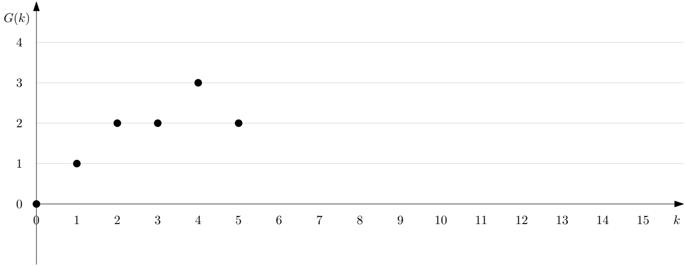

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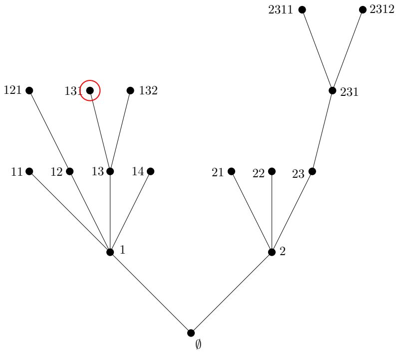

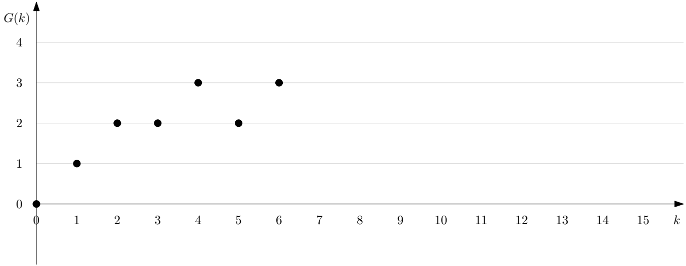

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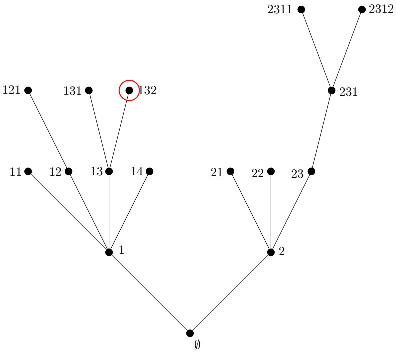

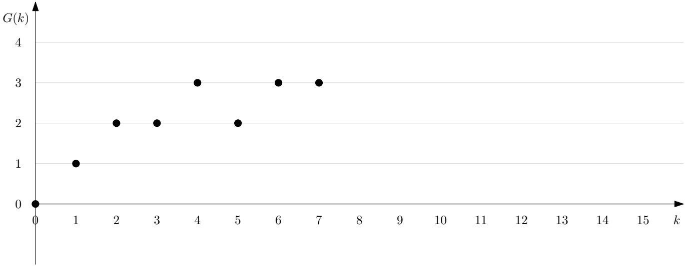

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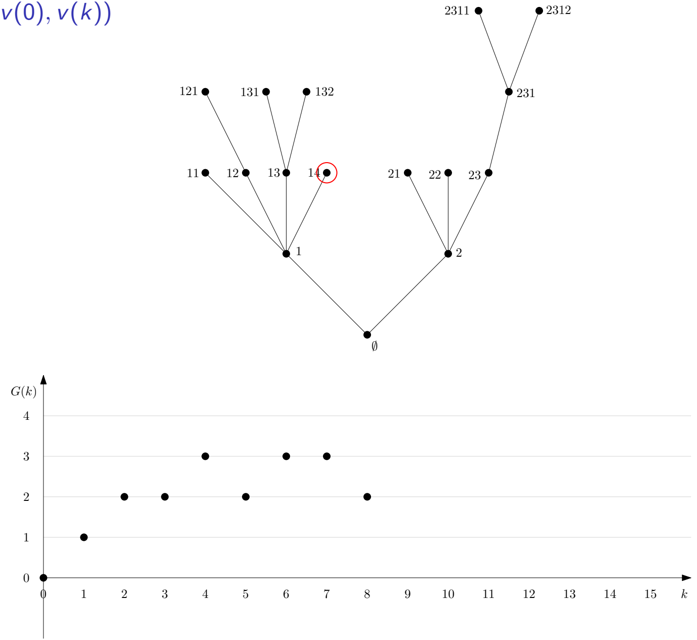

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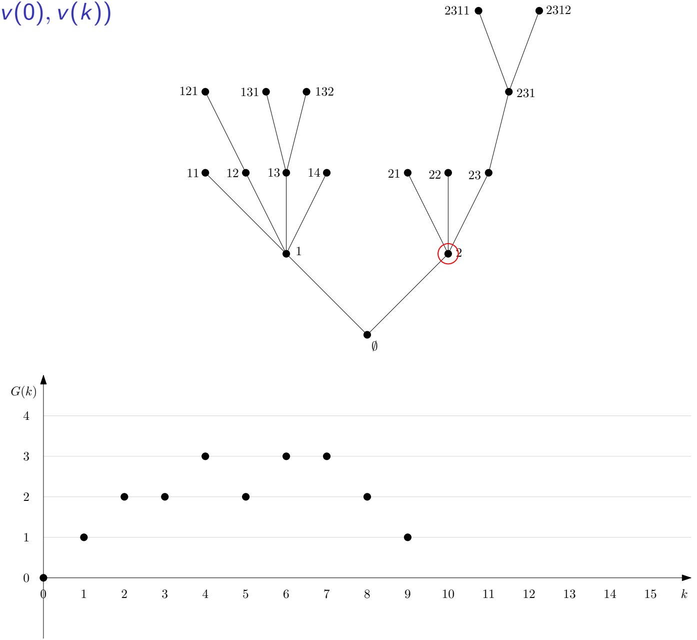

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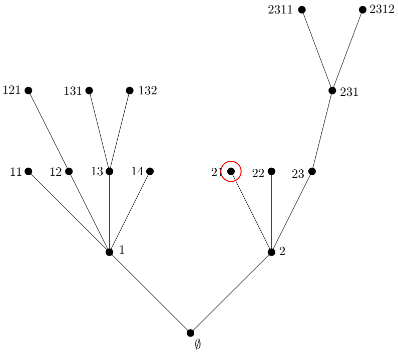

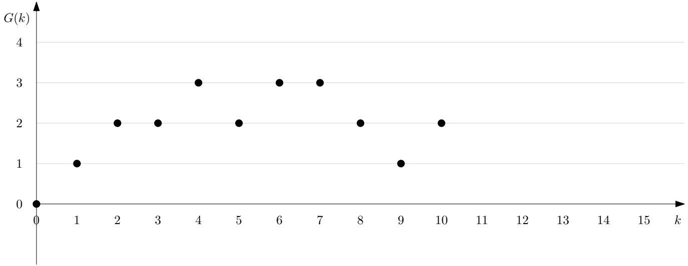

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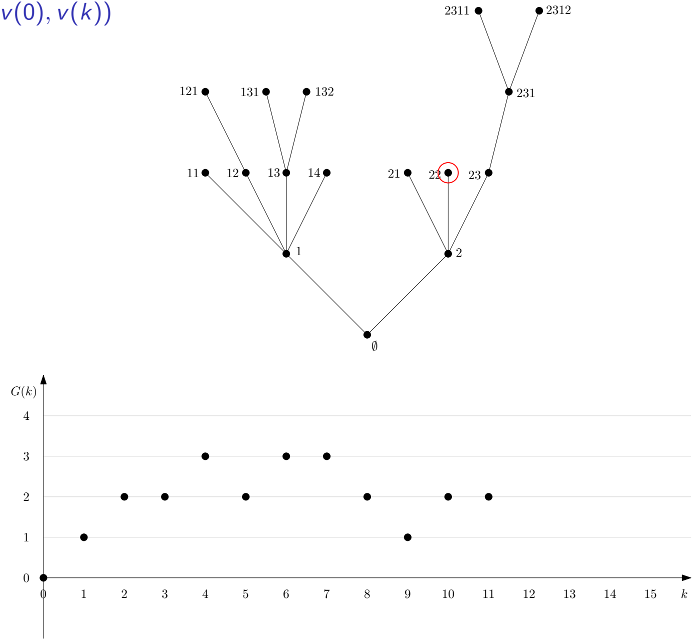

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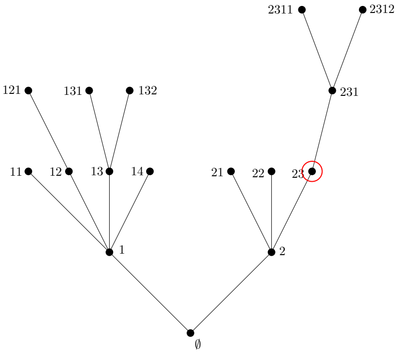

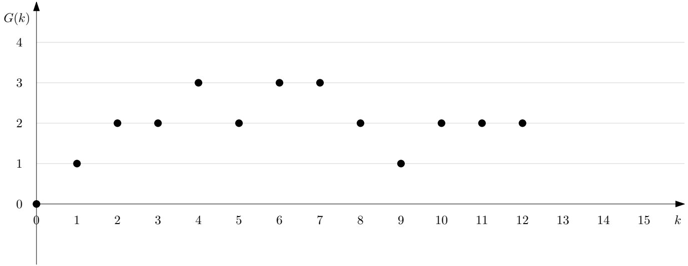

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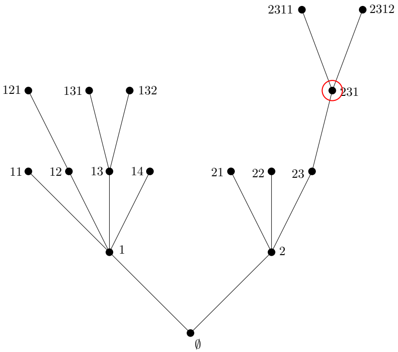

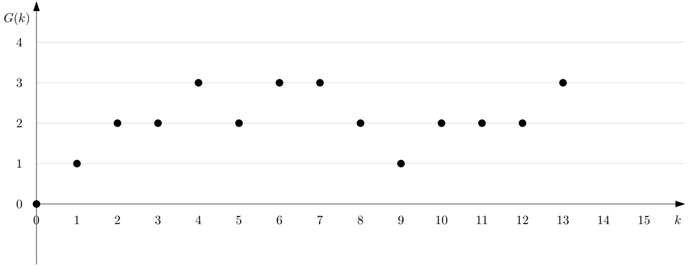

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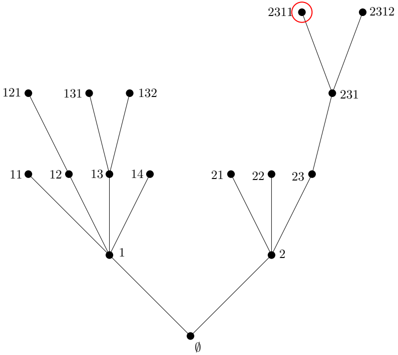

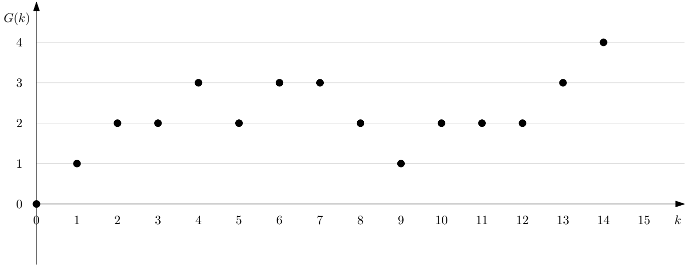

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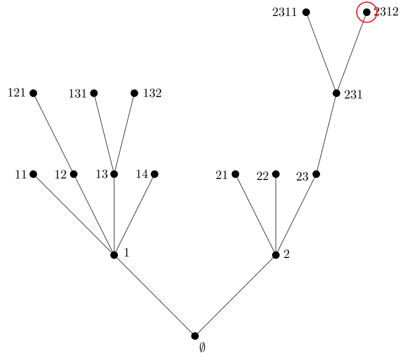

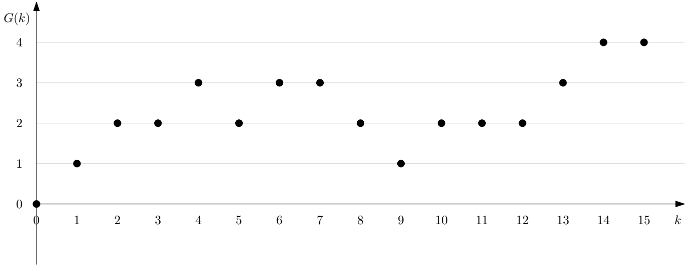

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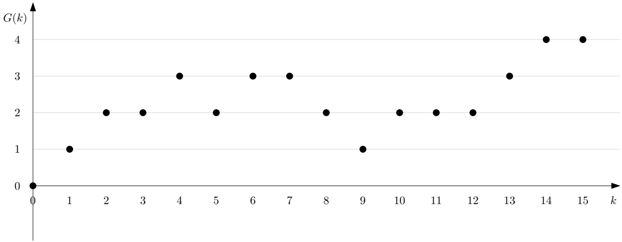
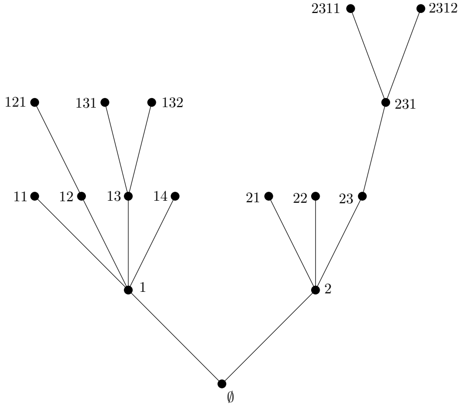
It's easy to recover the tree from its height process.



Depth-first walk

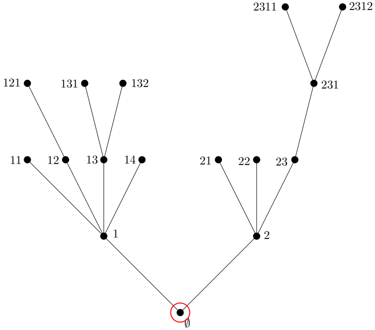
Let $C(k)$ be the number of children of $v(k)$, for $0 \leq k \leq n-1$, let $S(0) = 0$ and for $1 \leq k \leq n$,

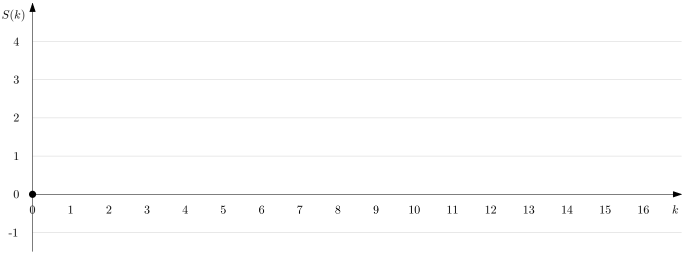
$$S(k) = \sum_{i=0}^{k-1} (C(i) - 1).$$



Depth-first walk

$S(k+1) = S(k) + C(k) - 1$





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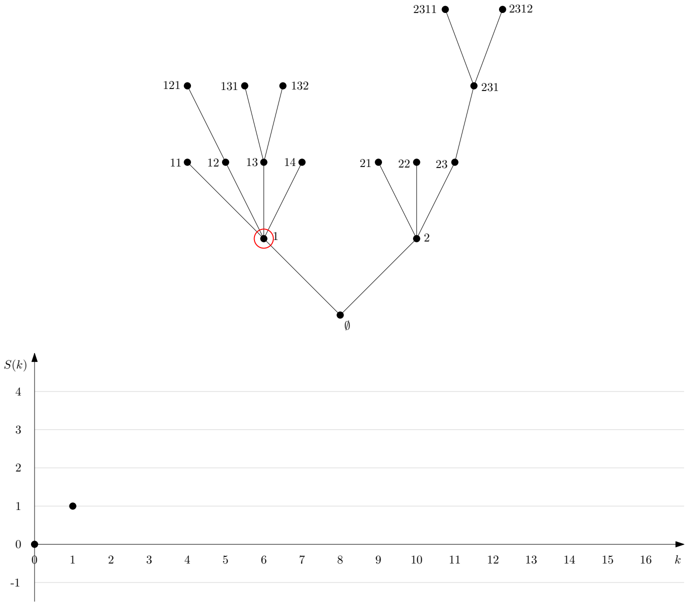

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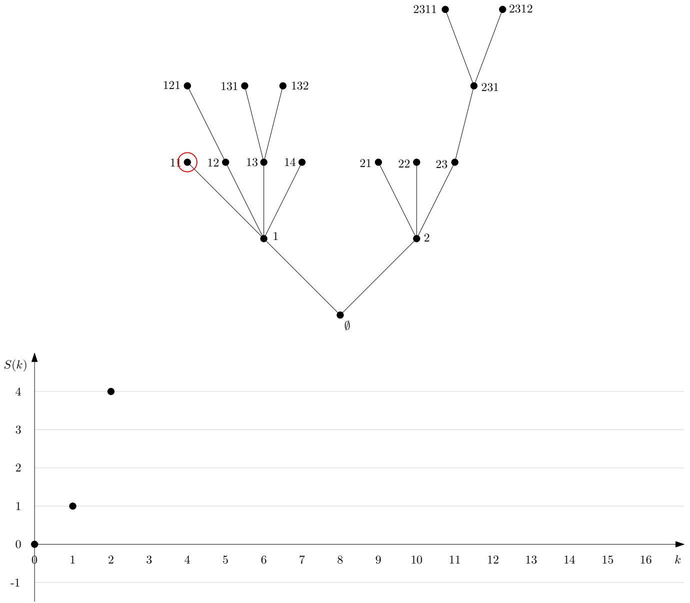

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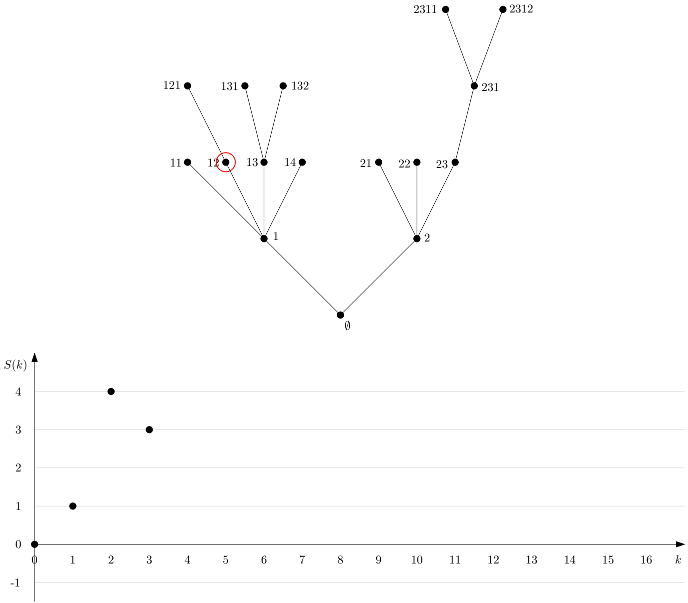

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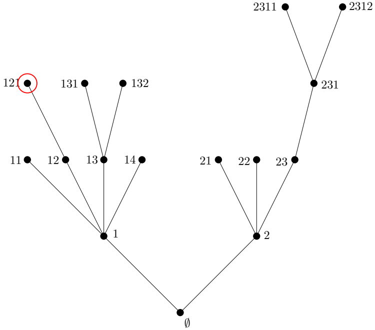

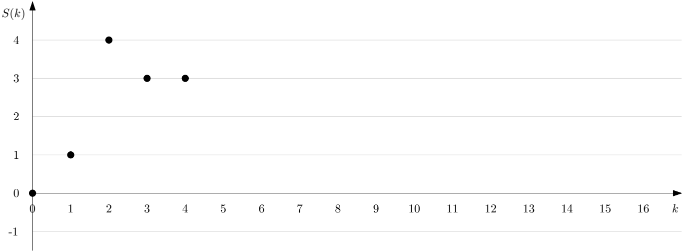

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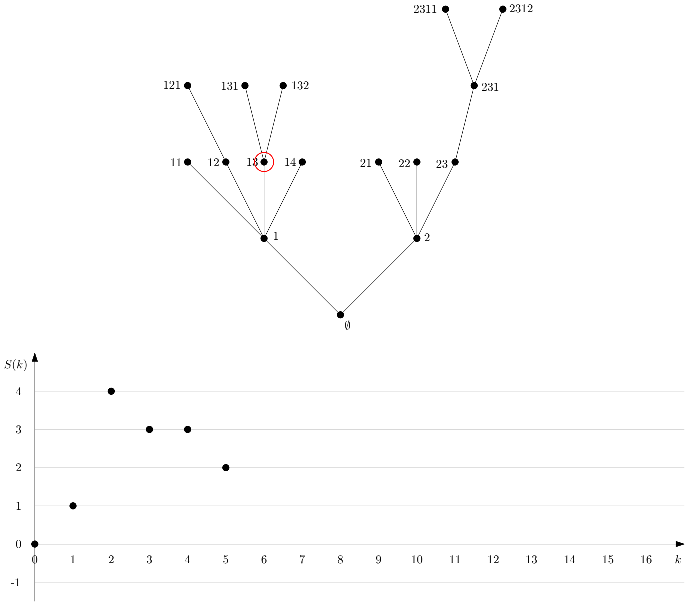

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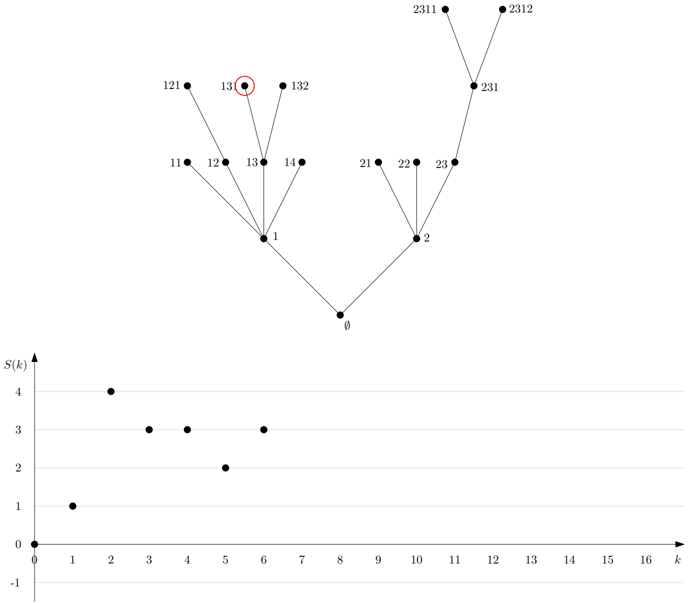

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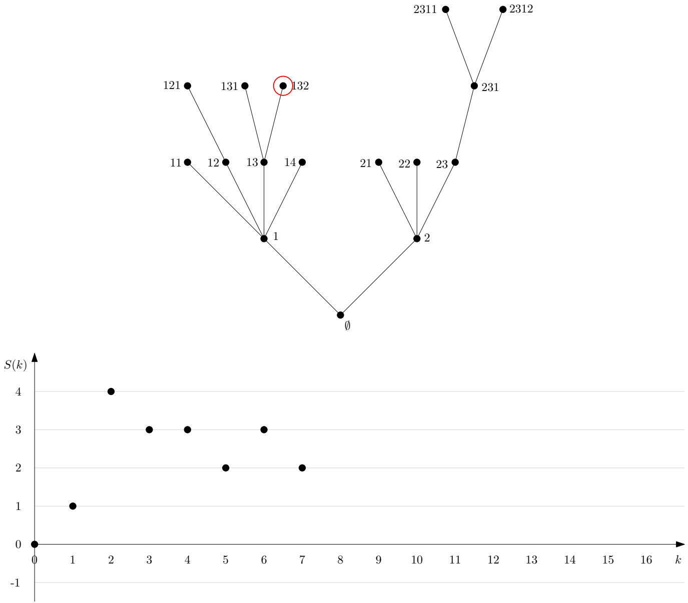

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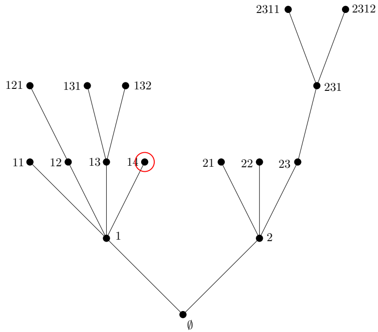

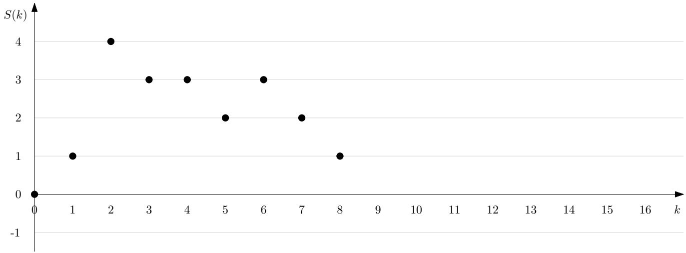

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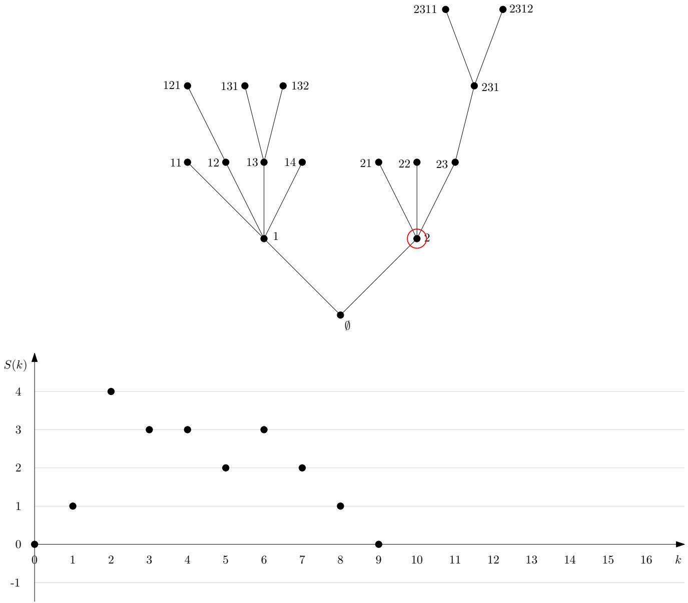

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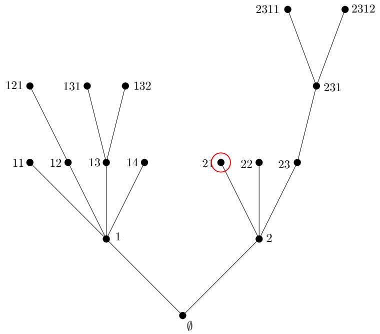

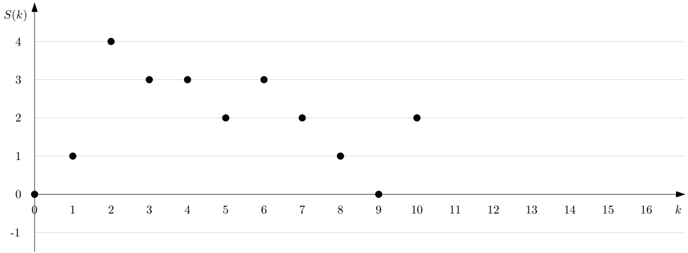

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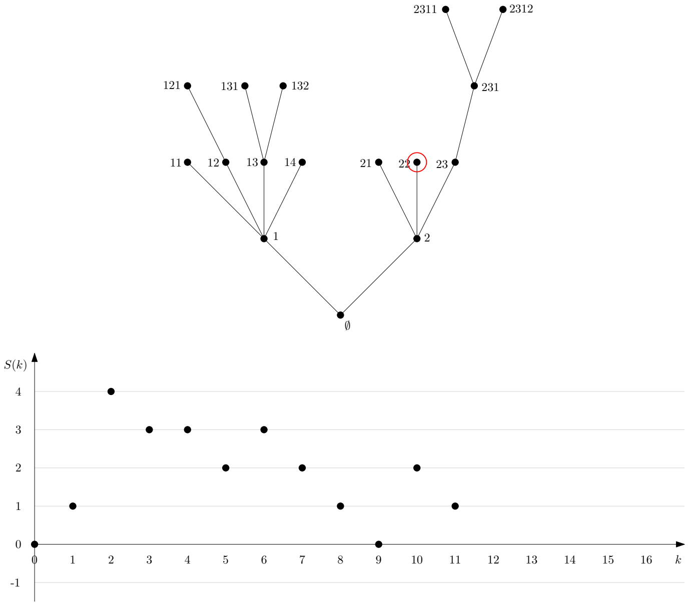

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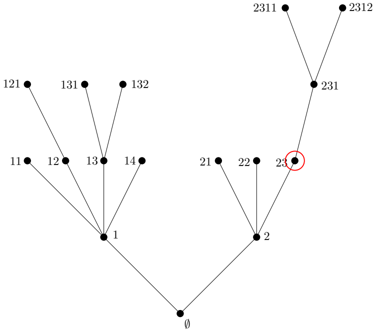

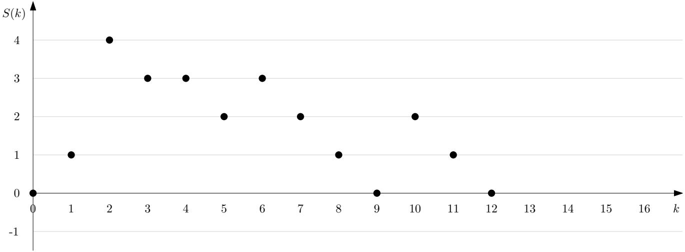

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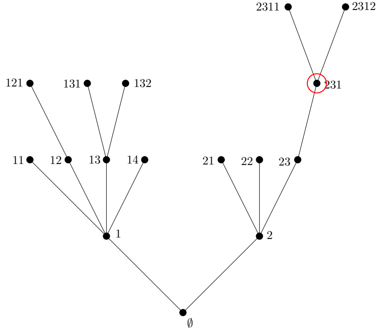

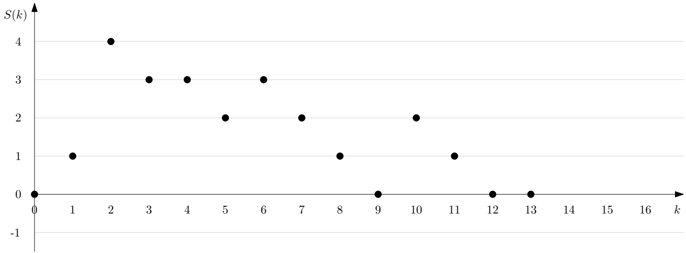

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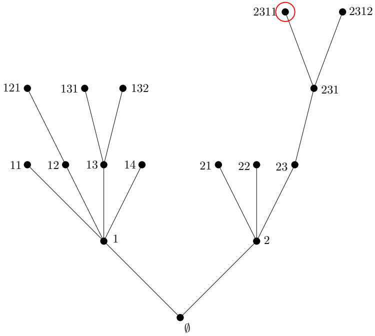

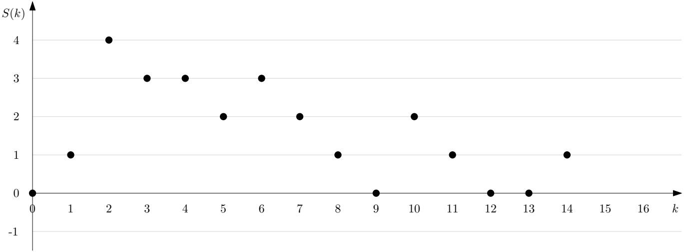

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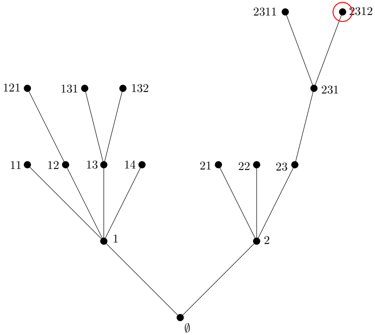

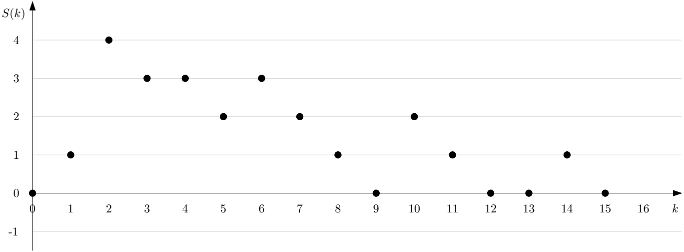

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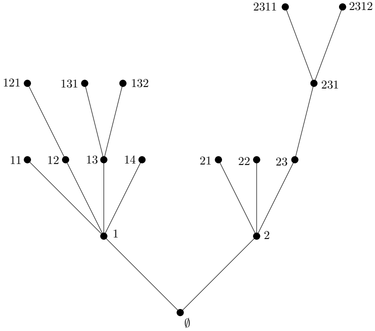

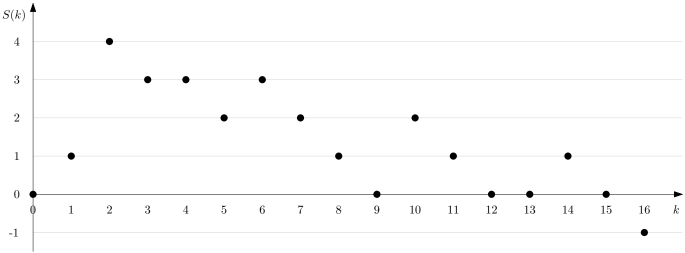

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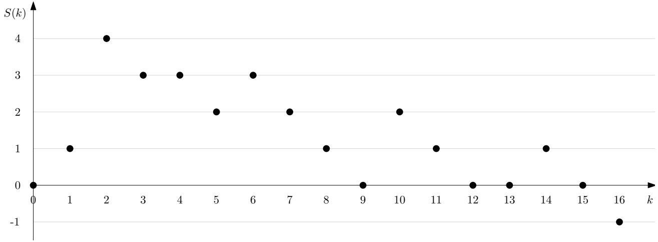

The depth-first walk counts vertices, apart from the current one, that we have seen, but not yet fully explored.

Depth-first walk

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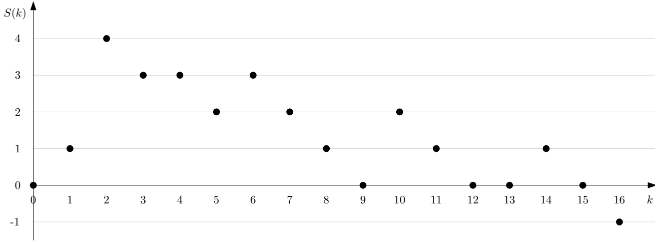

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Observe that the depth-first walk must hit -1 at step n, since $\sum_{i=0}^{n-1} C(i) = n - 1$ i.e. $\sum_{i=0}^{n-1}(C(i) - 1) = -1$.

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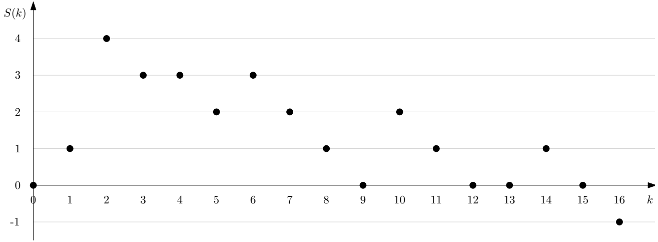

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Observe that the depth-first walk must hit -1 at step n, since $\sum_{i=0}^{n-1} C(i) = n-1$ i.e. $\sum_{i=0}^{n-1}(C(i)-1) = -1$. Moreover, it must be at 0 or above until then, since there must be a non-negative number of other nodes left to explore.

Height process and depth-first walk

The height process (and therefore the tree) may be recovered from the depth-first walk via

$$G(k) = \# \left\{ 0 \le j \le k-1 : S(j) = \min_{j \le \ell \le k} S(\ell) \right\}.$$

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Essential idea: whenever the depth-first walk enters a new subtree, it remains above its value at the start of the subtree until it leaves the subtree, when it goes one step lower. So instants j such that $S(j) = \min_{j \leq \ell \leq k} S(\ell)$ correspond to subtrees that we have entered but not yet finished exploring by the time we visit $v(k)$. But the number of such instants is the same as the generation of $v(k)$.

Galton–Watson forests

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Instead, consider a sequence of i.i.d. Galton–Watson trees. It is convenient to start the depth-first walk for the ith tree from $-i+1$, so that at the end of each tree the depth-first walk attains a new minimum. If we do this then defining

$$G(k) = \#\left\{0 \le j \le k-1 : S(j) = \min_{j \le \ell \le k} S(\ell)\right\}$$

as before yields a process which is at 0 every time we visit the root vertex of a component.

Galton–Watson forests

Since the numbers of children of the different vertices are i.i.d., $(S(k))_{k\geq 0}$ is a random walk with step-sizes $C(k)-1$, $k\geq 0$. Since $\mathbb{E}[C(0)]=1$, this random walk is centred. (In contrast, the law of the height process is much harder to describe.)

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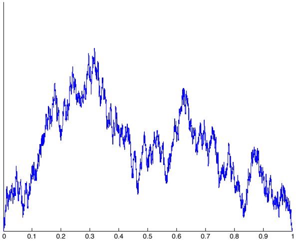
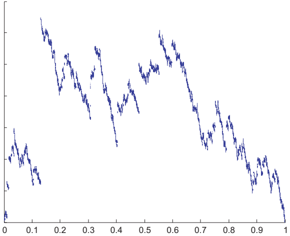
3. More general settings (with n-dependent offspring distributions) give rise to more general spectrally positive Lévy processes in the limit.

Interpretation

Recall that the depth-first walk attains a new minimum every time it starts exploring a new component. In the limit, the excursions above the running infimum should encode limiting "trees". The height process gives us a way to deal with them as metric spaces.





[Pictures by Igor Kortchemski]

Scaling limits

[Duquesne & Le Gall (2002)]

The height process is, however, more complicated. We have

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In the Brownian and stable cases, the height process is continuous.

Scaling limits

[Duquesne & Le Gall (2002)]

1. In the Brownian case, it turns out that

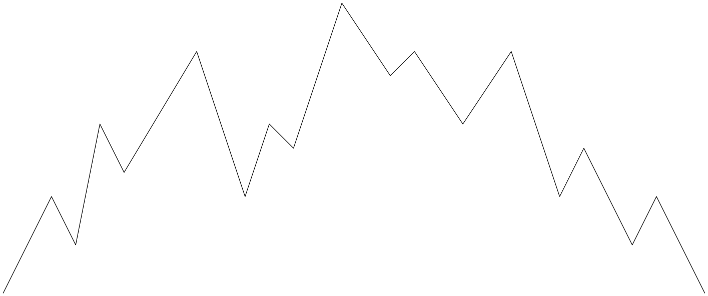
$$\frac{1}{\sqrt{n}}(G(\lfloor nt \rfloor), t \geq 0) \xrightarrow{d} \frac{2}{\sigma}(H_t, t \geq 0),$$

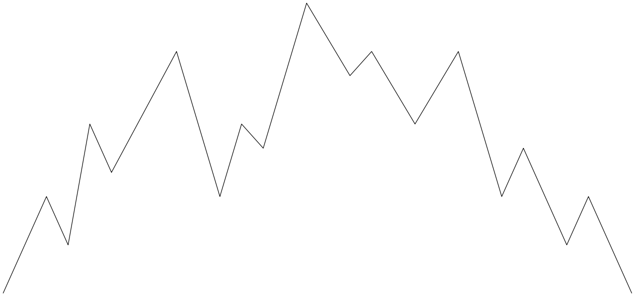
   where H_t is a reflected Brownian motion.

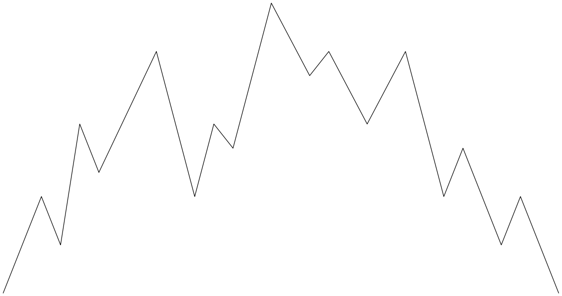
2. More generally, in the α-stable case, we get

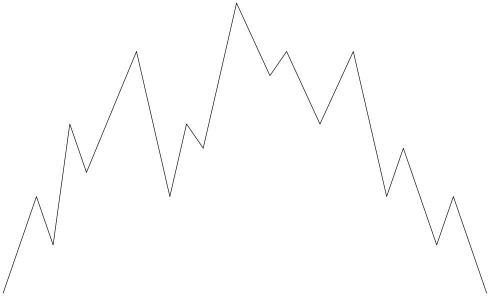
$$n^{-\frac{(\alpha-1)}{\alpha}}(G(\lfloor nt \rfloor), t \geq 0) \xrightarrow{d} C(H_t, t \geq 0).$$

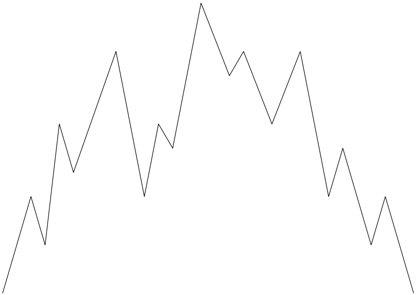
Idea: excursions of the limiting height process above 0 code limiting trees ($\mathbb{R}$-trees), the tallest of which have heights of order $n^{\frac{\alpha-1}{\alpha}}$, $\alpha \in (1, 2]$.

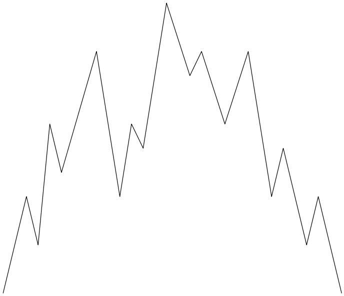


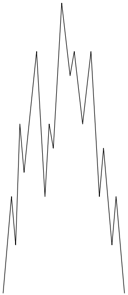


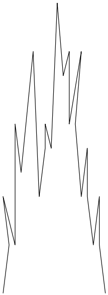


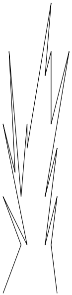


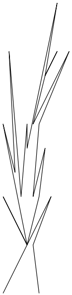


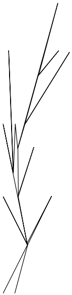


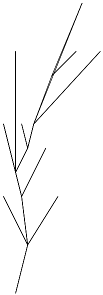


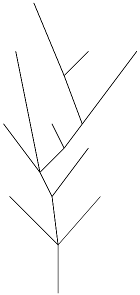


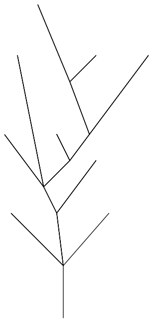






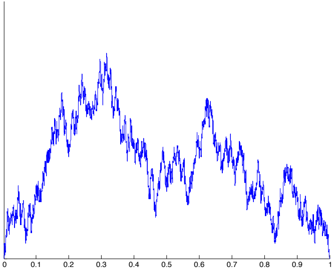


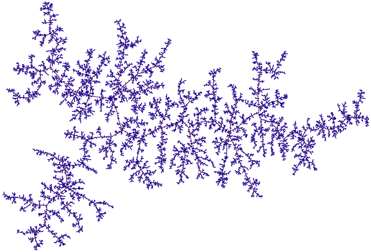




(Interpret distances vertically)

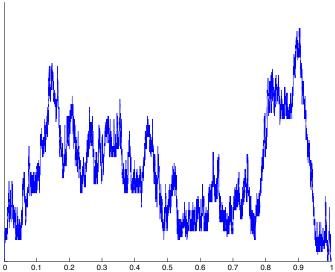
Brownian continuum random tree

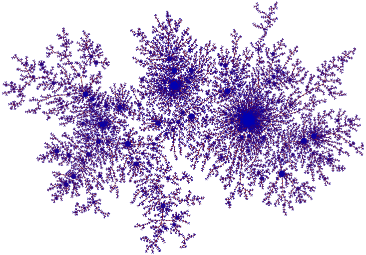


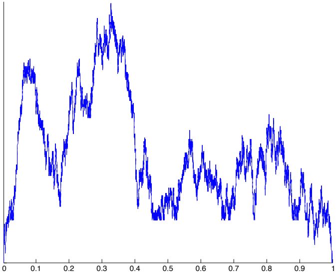


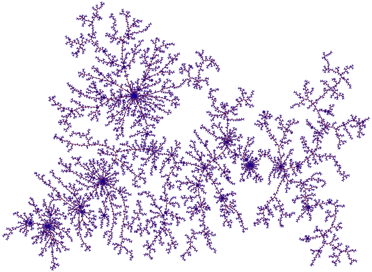
[Pictures by Igor Kortchemski]

α-stable trees ($\alpha = 1.1$ and $\alpha = 1.5$)









[Pictures by Igor Kortchemski]

PART II: RANDOM GRAPHS:

the Erdős–Rényi universality class

The Erdős–Rényi random graph

[Erdős & Rényi (1960)]

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Modern proofs of this phase transition essentially involve comparing the components to branching processes.

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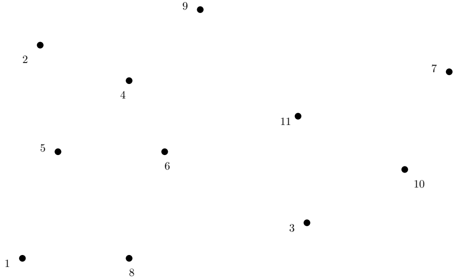

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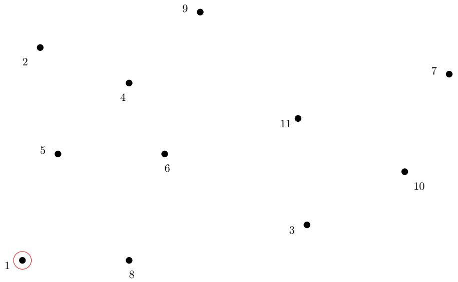

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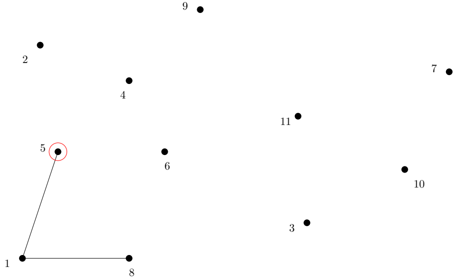

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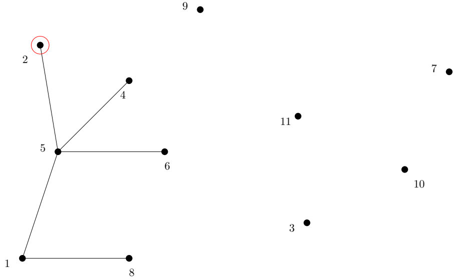

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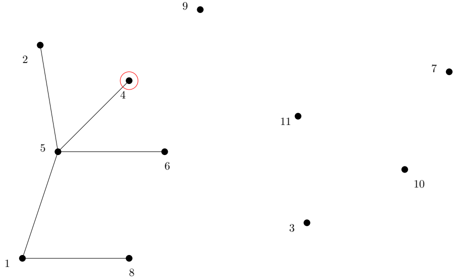

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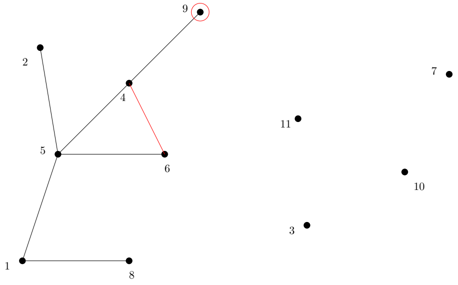

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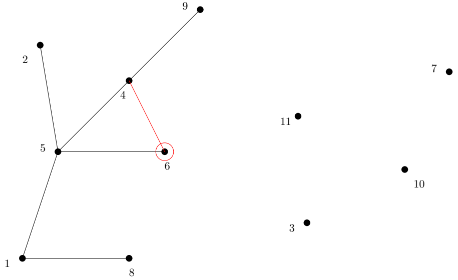

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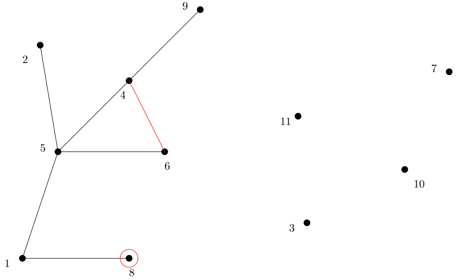

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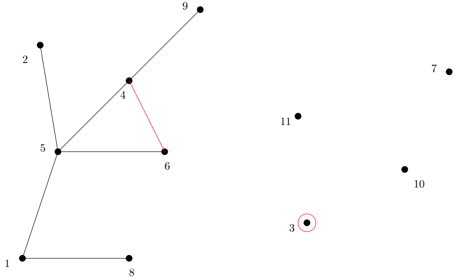

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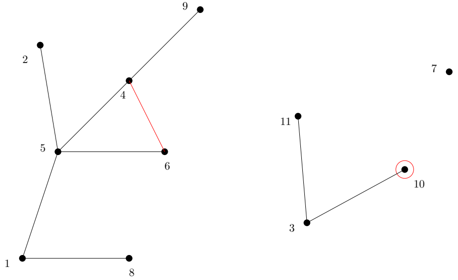

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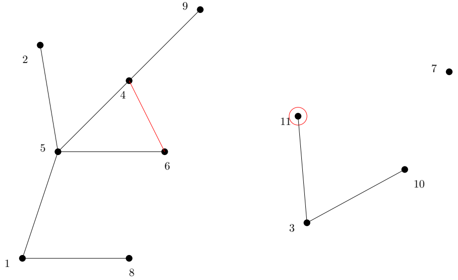

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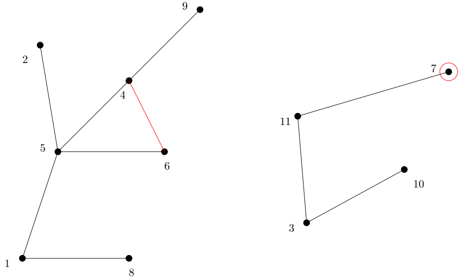

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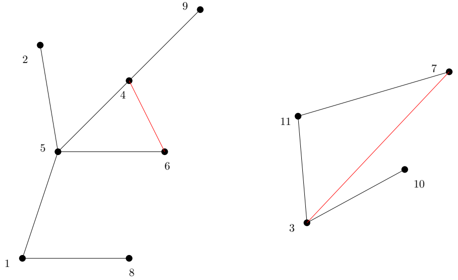

Depth-first walk

As before, let

$$S^n(k) = \sum_{i=0}^{k-1} (C^n(i) - 1), \qquad 0 \le k \le n,$$

where $C^n(i)$ is the number of children of the ith vertex explored in depth-first order.

Depth-first walk

As long as we have explored $o(n)$ vertices, it remains the case that the number of children of a vertex is approximately Po(1), although as we eat away at the vertices, there are fewer and fewer possible neighbours. This effect appears in the limit as a negative drift.

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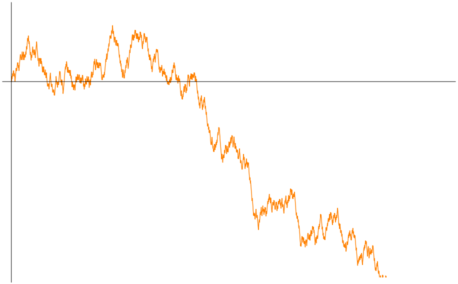
Theorem (Aldous (1997), breadth-first)

$$\frac{1}{n^{1/3}}\left(S^n(\lfloor tn^{2/3}\rfloor), t \geq 0\right) \xrightarrow{d} \left(B_t - \frac{t^2}{2}, t \geq 0\right).$$



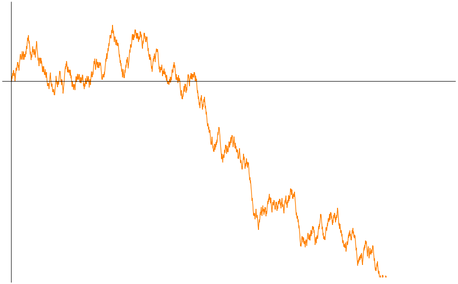
[Picture by Louigi Addario-Berry]

Component sizes and surplus edges

We start a new component every time we create a new minimum. Let

$$Z_t := B_t - \frac{t^2}{2} - \inf_{0 \le s \le t} \left(B_s - \frac{s^2}{2} \right), \qquad t \ge 0.$$

This represents the limiting rescaled number of vertices seen but not fully explored at time t.



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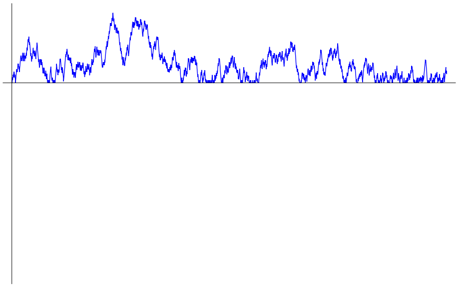
This represents the limiting rescaled number of vertices seen but not fully explored at time t. Every time Z hits 0, a new component begins.

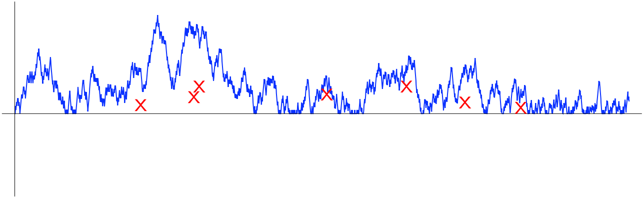


[Picture by Louigi Addario-Berry]

Component sizes and surplus edges

Aldous also showed that the edges forming cycles arise as a point process which in the limit is Poisson with intensity given by Z_t at time t.

We may think of the Poisson points as occurring with intensity 1 in the area under the graph of Z.



[Picture by Louigi Addario-Berry]

Component sizes and surplus edges

Let $\mathbf{C}^n = (C_1^n, C_2^n, \ldots)$ be the sizes of the components, listed in decreasing order, and $\mathbf{S}^n = (S_1^n, S_n^2, \ldots)$ the corresponding numbers of surplus edges.

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Theorem (Aldous (1997))

As $n \to \infty$,

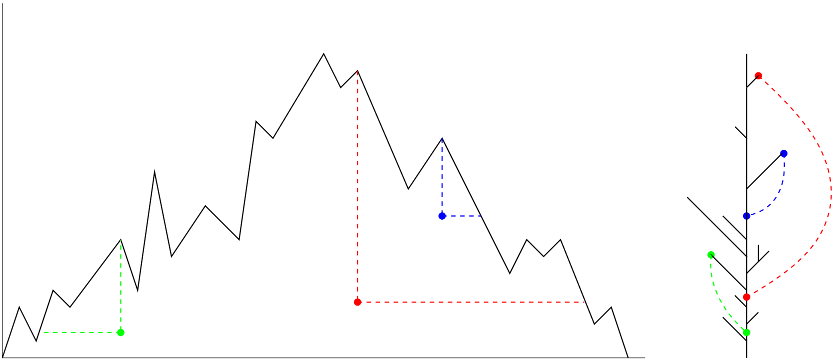
$$\left(n^{-2/3}\mathbf{C}^n, \mathbf{S}^n\right) \xrightarrow{d} (\mathbf{C}, \mathbf{S}),$$

where the convergence of the component sizes is in $\ell_2^{\downarrow}$.

Metric space scaling limit

[Addario-Berry, Broutin & G. (2012)]

The excursions encode spanning subtrees, and the points of the Poisson process tell us where to make vertex-identifications.



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$$\mathbb{E}[f(\tilde{e})] = \frac{\mathbb{E}\left[f(e) \exp\left(\int_0^x e(u) du\right)\right]}{\mathbb{E}\left[\exp\left(\int_0^x e(u) du\right)\right]},$$

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where e is a Brownian excursion of length x.

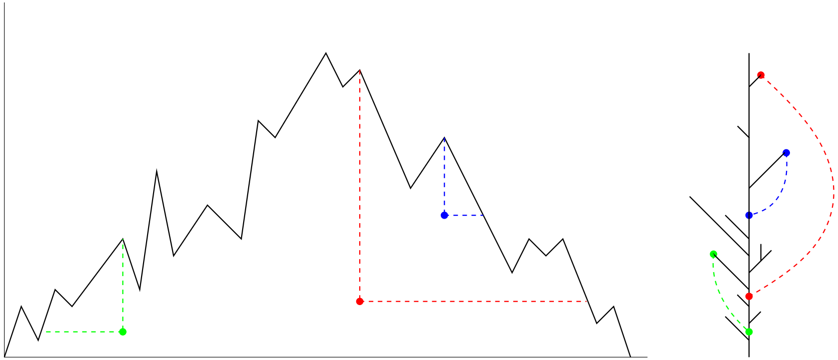
Conditionally on $\tilde{e}$, we get a Poisson number of vertex-identifications with mean

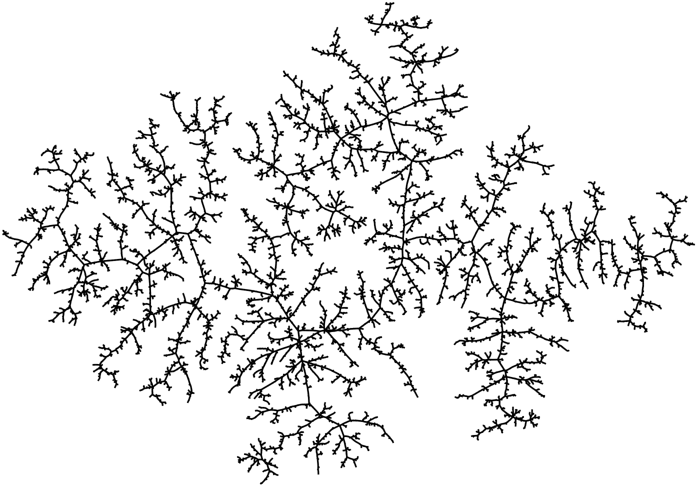
$$\int_0^x \tilde{e}(u)du.$$

Each identifies a random leaf with a uniformly-chosen point down the backbone to the root.

Metric space scaling limit

[Addario-Berry, Broutin & G. (2012)]





[Picture by Nicolas Broutin]

Universality

The same objects have been shown to occur as the scaling limit in a variety of settings.

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Component sizes (and in some cases surpluses):

- Critical percolation on random regular graphs: Nachmias & Peres (2010)
- Critical random graphs with given degree sequence (with finite third moment): Riordan (2012), Joseph (2014)
- Critical inhomogeneous random graphs (weights with finite third moment): Aldous (1997), Turova (2013), Bhamidi, van der Hofstad & van Leeuwaarden (2010)
- Achlioptas processes with bounded size rules at criticality: Bhamidi, Budhiraja & X. Wang (2013)

Universality

Metric structure:

- Very general, encompassing all of the above models; framework based on scaling exponents and approximation by the multiplicative coalescent: Bhamidi, Sen & X. Wang (2014+), Bhamidi, Broutin, Sen & X. Wang (2014+)

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See Shankar Bhamidi's talk, *Continuum scaling limits of critical inhomogeneous random graph models,* on Thursday afternoon in the **Interacting particle systems and their scaling limits** session.

Conjectural Erdős–Rényi universality class

The Erdős–Rényi random graph can be thought of as a mean-field model for percolation on a finite graph. It is conjectured that for a wide variety of finite base graphs G_n which are sufficiently "high dimensional", although the percolation critical point will be model-dependent, the behaviour in the vicinity of that critical point should essentially be the same as in the Erdős–Rényi model.

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Evidence in the setting of the hypercube and other high-dimensional tori: Borgs, Chayes, van der Hofstad, Slade & Spencer (2005a,b), Heydenreich & van der Hofstad (2011), van der Hofstad & Sapozhnikov (2014).

Outside the Erdős–Rényi universality class

The Erdős–Rényi random graph is a poor model for many real-world networks. In particular, there is a lot of interest in modelling situations where we observe power-law degree distributions.

Outside the Erdős–Rényi universality class

There has been much recent work on a particular model for inhomogeneous random graphs (the Norros–Reittu model) with parameters chosen to give power-law degrees. Analogous results to those we obtained in the Erdős–Rényi setting have been developed in a series of papers by Bhamidi, van der Hofstad, van Leeuwaarden and Sen, and in work in progress by Broutin, Duquesne & M. Wang.

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The limit spaces they obtain are certain tilted inhomogeneous continuum random trees [Aldous & Pitman (2000)] again with a finite number of additional vertex-identifications. The approach via the height process used for the Erdős–Rényi random graph doesn't work here, since there is currently no convergence result for the height processes in this context.

PART III: RANDOM GRAPHS:

i.i.d. degrees with power-law tails

[Based on work in progress with Guillaume Conchon-Kerjan (ENS)]

Random graphs with given degrees

Consider a graph G_n chosen uniformly at random from those such that the vertex set is $\{1, 2, \ldots, n\}$ and vertex i has degree (number of neighbours) d_i.

Configuration model

[Bender & Canfield (1978), Bollobás (1980), Wormald (1978), ...]

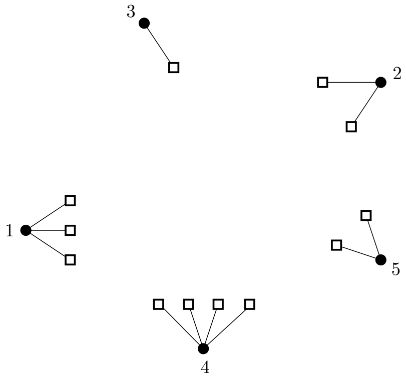
Standard method for generating a (multi)graph on n vertices with given degrees $d_1, d_2, \ldots, d_n$.

Suppose $d_i \geq 1$ for all $1 \leq i \leq n$ and $\ell_n = \sum_{i=1}^{n} d_i$ is even.

Assign d_i "half-edges" or "stubs" to the vertex labelled i. Number the stubs in an arbitrary way from 1 to ℓ_n. Now pair the half-edges uniformly at random to form edges.

Configuration model

Example: $n = 5$ and $d_1 = 3$, $d_2 = 2$, $d_3 = 1$, $d_4 = 4$, $d_5 = 2$.



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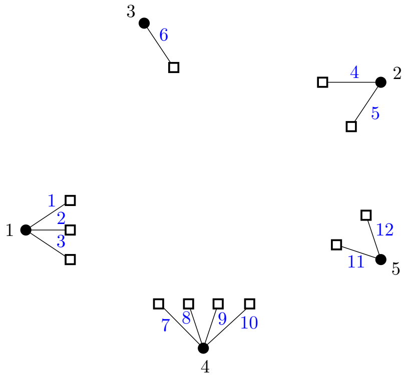

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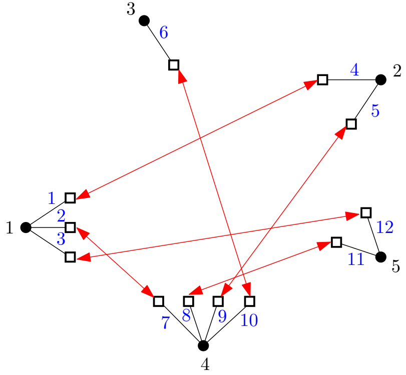

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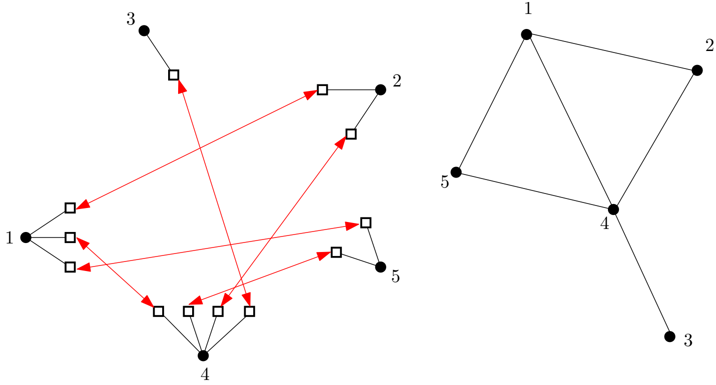

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This procedure can give rise to loops or multiple edges, in which case we have a multigraph. But if we condition the graph to have no loops or multiple edges (to be simple), then it is uniformly chosen from the set of graphs with these degrees.

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(It's not always the case that a particular degree sequence with even sum can give a simple graph, so this conditioning may not always be valid. This will not be problematic in the context we consider.)

Configuration model with i.i.d. degrees

Suppose that we have i.i.d. random degrees, $D_1, D_2, \ldots, D_n$ having finite variance, and let $\gamma = \mathbb{E}\left[D(D-1)\right]/\mathbb{E}\left[D\right]$.

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$$\mathbb{P}(G_n \text{ is simple}) \to \exp(-\gamma/2 - \gamma^2/4).$$

Important point: we can generate the matching of the half-edges edge by edge, in any order that is convenient. In particular, rather than first sampling the graph and then exploring it, we will find it useful to generate the graph step-by-step as we explore it.

Configuration model with i.i.d. degrees

[Molloy & Reed (1995)]

Recall that $\gamma = \mathbb{E}[D(D-1)] / \mathbb{E}[D]$. The critical point for the emergence of a giant component is $\gamma = 1$.

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$$\mathbb{P}(D^* = k) = \frac{k\mathbb{P}(D = k)}{\mathbb{E}[D]}, \quad k \geq 1.$$

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So the "offspring distribution" to which we should compare is the law of $D^* - 1$ which has

$$\mathbb{E}[D^* - 1] = \frac{\mathbb{E}\left[D^2\right]}{\mathbb{E}[D]} - 1 = \frac{\mathbb{E}[D(D-1)]}{\mathbb{E}[D]} = \gamma.$$

Power-law tails

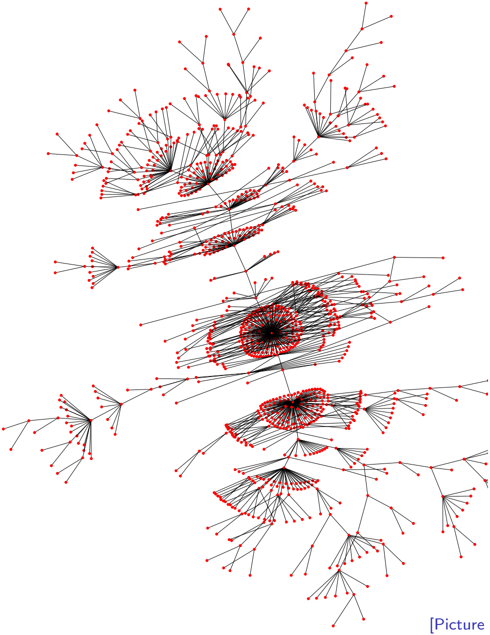
[Joseph (2014)]

We have i.i.d. degrees $D_1, D_2, \ldots, D_n$ with law ν such that

1. $\mathbb{P}(D_1 \geq 1) = 1$
2. $\gamma = \mathbb{E}[D_1(D_1 - 1)] / \mathbb{E}[D_1] = 1$
3. $\mathbb{P}(D_1 = k) \sim ck^{-(\alpha+2)}$ as $k \to \infty$, for some $c > 0$, $\alpha \in (1, 2)$.

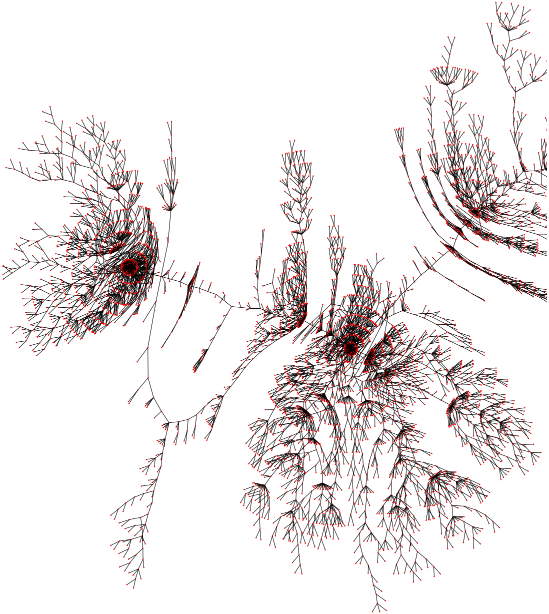
Write $\mu = \mathbb{E}[D_1]$ (our conditions imply that $\mu \in (1, 2)$).

$\alpha = 1.2$



[Picture by Delphin Sénizergues]

$\alpha = 1.5$



[Picture by Delphin Sénizergues]

$\alpha = 1.8$

[Picture by Delphin Sénizergues]

Depth-first exploration

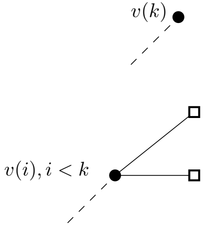
[Riordan (2012); Joseph (2014)]

Sample the degrees $D_1, D_2, \ldots, D_n$ and then start from a vertex $v(0)$ chosen with probability proportional to its degree.

For $k \geq 0$, proceed as follows.

Depth-first exploration

[Riordan (2012); Joseph (2014)]



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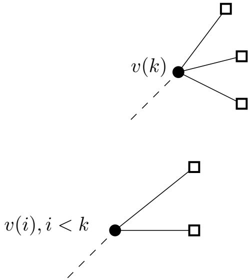

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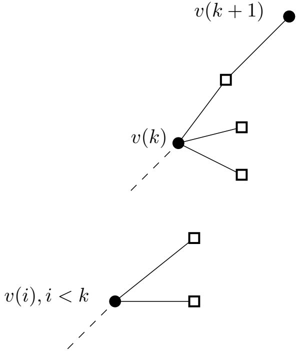

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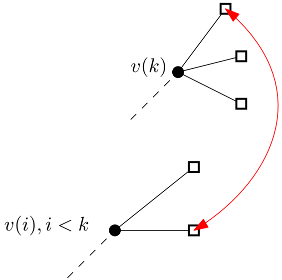

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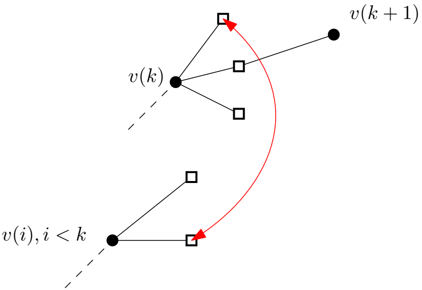

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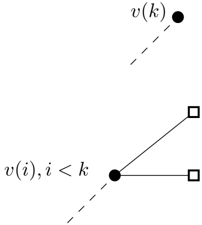

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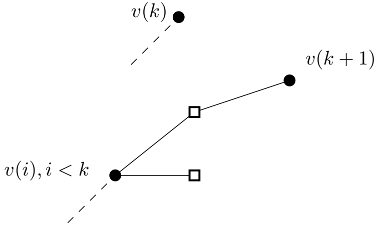

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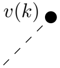

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Important point: in any case, we see the vertices in size-biased order of degree: $(\hat{D}_1^n, \hat{D}_2^n, \ldots, \hat{D}_n^n)$.

Approximate depth-first walk

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This is an approximation in two ways:

1. for the vertex at the start of a component, the number of children is actually $\hat{D}_i^n$ rather than $\hat{D}_i^n - 1$;
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2. it ignores the possibility of surplus edges.

Neither is problematic in the limit.

Indeed, it is possible to show that there are only $O(1)$ surplus edges in the first $O(n^{\alpha/(\alpha+1)})$ steps.

Approximate depth-first walk

Theorem (Joseph (2014))

$$n^{-1/(\alpha+1)}\left(\tilde{S}^n(\lfloor tn^{\alpha/(\alpha+1)}\rfloor), t \geq 0\right) \xrightarrow{d} (\tilde{L}_t, t \geq 0),$$

where $\tilde{L}$ is the process with independent increments characterised by its Laplace transform

$$\mathbb{E}\left[\exp(-\lambda \tilde{L}_t)\right]$$

$$= \exp\left(\int_0^t ds \int_0^\infty dx(e^{-\lambda x} - 1 + \lambda x)\frac{c}{\mu x^{\alpha+1}}e^{-xs/\mu} - \lambda C_\alpha \frac{t^\alpha}{\mu^\alpha}\right),$$

where $C_\alpha = \frac{c\Gamma(2-\alpha)}{\alpha(\alpha-1)}$.

Component sizes

Let $\mathbf{C}^n = (C_1^n, C_2^n, \ldots)$ be the ordered component sizes of the multigraph G_n, and let $\mathbf{C} = (C_1, C_2, \ldots)$ be the ordered lengths of excursions of $\tilde{L}$ above its running infimum.

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as $n \to \infty$, in $\ell_2^{\downarrow}$.

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as $n \to \infty$, *in* $\ell_2^{\downarrow}$.

Note: this is the same scaling as in [Bhamidi, van der Hofstad & van Leeuwaarden (2012)], but a different limit.

Absolute continuity relations

[Conchon-Kerjan & G. (in progress)]

Let $D_1^*, D_2^*, \ldots$ be i.i.d. with law $k\nu_k/\mu$, $k \geq 1$ (the true size-biased degree distribution) and let $S(0) = 0$ and

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$$\mathbb{E}\left[\exp(-\lambda L_t)\right] = \exp\left(t \int_0^\infty dx (e^{-\lambda x} - 1 + \lambda x) \frac{c}{\mu x^{\alpha+1}}\right)$$
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L encodes a forest of stable trees.

Absolute continuity relations

Proposition

For every $t \geq 0$, we have the following absolute continuity relation: for every suitable test-functional F,

$$\mathbb{E}\left[F\left(\tilde{L}_s, 0 \leq s \leq t\right)\right]$$
$$= \mathbb{E}\left[\exp\left(-\frac{1}{\mu}\int_0^t s dL_s - C_\alpha \frac{t^{\alpha+1}}{(\alpha+1)\mu^{\alpha+1}}\right) F(L_s, 0 \leq s \leq t)\right].$$

Absolute continuity relations

There is also a discrete analogue: for $m < n$,

$$\mathbb{E}\left[f(\hat{D}_1^n, \hat{D}_2^n, \ldots, \hat{D}_m^n)\right] = \mathbb{E}\left[\phi_m^n(D_1^*, D_2^*, \ldots, D_m^*) f(D_1^*, D_2^*, \ldots, D_m^*)\right]$$

where for $m = \lfloor tn^{\alpha/(\alpha+1)} \rfloor$,

$$\begin{aligned}&\phi_m^n(D_1^*, D_2^*, \ldots, D_m^*)\\ &\qquad \xrightarrow{d} \exp\left(-\frac{1}{\mu}\int_0^t s dL_s - C_\alpha \frac{t^{\alpha+1}}{(\alpha+1)\mu^{\alpha+1}}\right).\end{aligned}$$

Height processes

Let

$$\tilde{G}^n(k) = \# \left\{ 0 \leq j \leq k-1 : \tilde{S}^n(j) = \min_{j \leq \ell \leq k} \tilde{S}^n(\ell) \right\}$$

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and define a height process $\tilde{H}$ via

$$\mathbb{E}\left[f(\tilde{L}_u, \tilde{H}_u, 0 \leq u \leq t) \right]$$

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where L and H are a spectrally positive α-stable Lévy process and the corresponding height process, respectively.

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where L and H are a spectrally positive α-stable Lévy process and the corresponding height process, respectively. Using Duquesne & Le Gall's theorem we can considerably strengthen Joseph's result:

Theorem

$$\left(n^{-\frac{1}{\alpha+1}}\tilde{S}^n(\lfloor un^{\alpha/(\alpha+1)}\rfloor), n^{-\frac{\alpha-1}{\alpha+1}}\tilde{G}^n(\lfloor un^{\alpha/(\alpha+1)}\rfloor), 0 \le u \le t\right)$$
$$\xrightarrow{d} \left(\tilde{L}_u, \tilde{H}_u, 0 \le u \le t\right).$$

Metric space scaling limit: the stable graph

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The change of measure acts on the excursions of the Lévy process to give that an excursion of length x of $\tilde{L}$ above its infimum is such that

$$\mathbb{E}\left[f(\tilde{e})\right] = \frac{\mathbb{E}\left[f(e)\exp\left(\frac{1}{\mu}\int_0^x e(u)du\right)\right]}{\mathbb{E}\left[\exp\left(\frac{1}{\mu}\int_0^x e(u)du\right)\right]},$$

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(Recall that the random quantity in the exponential martingale is

$$-\frac{1}{\mu}\int_0^t s dL_s = -\frac{tL_t}{\mu} + \frac{1}{\mu}\int_0^t L_s ds$$

and note that $L_t = 0$ at the beginning and end of each excursion.)

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Metric space scaling limit: the stable graph

Neither multiple edges nor loops occur until $>> n^{\alpha/(\alpha+1)}$ steps of the exploration have occurred, so conditioning the graph to be simple does not affect the distribution of the large components.

Metric space scaling limit: the stable graph

The surplus edges can again be shown to occur as a Poisson point process with unit intensity in the area under the graph of

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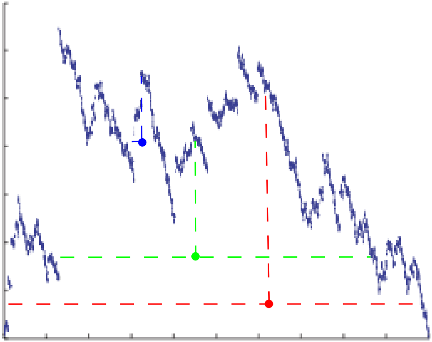
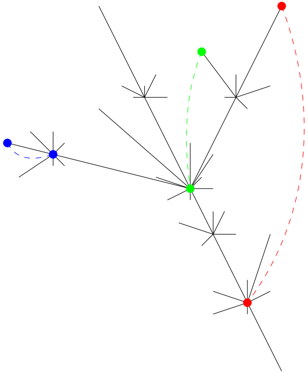
In the limit, the vertex-identifications are from leaves to hubs (branch-points of infinite degree).





Consequences

Distributional and geometric information about the limiting spaces may be deduced from knowledge of the stable trees.

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For example, the Hausdorff dimension of the limiting metric spaces is $\alpha/(\alpha - 1)$ almost surely.

Perspectives: line-breaking constructions

There is a beautiful construction of the Brownian CRT via line-breaking, due to Aldous. In [Addario-Berry, Broutin & G. (2010)], we showed that a closely related line-breaking construction can be used to build a limit component in the Erdős–Rényi random graph. In [G. & Haas (2015)], we proved a (more complicated) line-breaking construction for the stable trees. I expect that there will be a related construction of the components of the stable graph.

Perspectives: generalisations

The absolute continuity relation holds for a broad class of spectrally positive Lévy processes which may be used to encode a forest, which suggests that these results should be generalisable beyond the stable setting.

Open problem

How can one relate the limits obtained by Bhamidi, van der Hofstad, van Leeuwaarden and Sen in the setting of the Norros-Reittu model to the stable graph? Can one obtain the stable graph by averaging?

Thank you for listening!